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On the construction of shape preserving taper curves

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There exists an algorithm for construction interpolating quadratic splines which preserves the monotony of the data. The taper curves formed with this algorithm, QO-splines, have many good qualities when a sufficient number of measured diameters of a tree is available. In fact, they may even be superior to certain shape preserving taper curves, MR-splines. This algorithm can be modified to preserve also the shape of the data. In the present paper the quality of taper curves constructed by a new shape preserving form of the algorithm is examined. For this purpose taper curves are formed for different sets of measurements and their properties are compared with the ones of QO-splines and MR-splines. The results indicate that these new shape preserving taper curves are in general better than QO-splines and MR-splines even if the differences may be small in many cases. The superiority is the clearer the less measurements are available.

On olemassa algoritmi, jolla voidaan muodostaa interpoloivia neliöllisiä splinifunktioita siten, että mittausaineiston monotonisuus säilyy. Tällä algoritmilla muodostetuilla runkokäyrillä, QO-splineillä, on monia hyviä ominaisuuksia, kun puusta on saatavilla useita mitattuja läpimittoja. Itse asiassa ne saattavat jopa olla parempia kuin tietyt muodon säilyttävät runkokäyrät, MR-splinit. Algoritmi voidaan muuntaa muotoon, jossa myös mittausaineiston muoto säilyy. Tässä työssä tutkitaan algoritmin uudella muodon säilyttävällä versiolla saatuja runkokäyriä. Näitä käyriä muodostetaan erilaisten mittausten perusteella ja käyrien ominaisuuksia verrataan QO- ja MR-splinien vastaaviin ominaisuuksiin. Osoittautuu, että nämä uudet muodon säilyttävät runkokäyrät ovat yleensä parempia kuin QO- ja MR-splinit joskin erot saattavat usein olla pieniä. Paremmuus on sitä selvempää mitä vähemmän mittauksia on käytettävissä.

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1 Introduction

A good taper curve should give a mathematical model of a tree stem which is able to produce the essential properties of the stem. In addition of the volume and the lumber assortment also the shape of the stem is in many cases important. Lahtinen (1988) has considered a situation where a set of diameters has been measured from relative heights of a tree. He has shown that a reliable taper curve can be formed by using quadratic splines so that it reproduces the measurements and preserves the monotony of the stem as presented by the broken line connecting the measured points used in the construction. Such a taper curve is said to *preserve the monotony*.

The definition and basic properties of spline functions are to be found for instance in the book of De Boor (1978). The interpolating quadratic spline we are interested in consists of polynomial pieces of degree at most two. The pieces are joined at points called *breakpoints* so that the resulting function is continuously differentiable. Every interpolating point is a breakpoint. The interpolating quadratic spline is theoretically able to preserve locally both the monotony and the convexity of the data as explained by Lahtinen (1988). The latter property means that the spline is on every interval convex or concave in the same way as the broken line connecting the measured points. Such a monotony preserving spline is said to *preserve the shape*.

The essential feature of the construction of a monotony or shape preserving spline is to choose the first derivatives of the spline at interpolating points as parameters and to add some additional breakpoints between interpolating points. These breakpoints are also used as parameters. A construction of such a shape preserving spline has been presented for instance by McAllister and Roulier (1981) as a Fortran-program.

In principle a shape preserving spline should be superior to a monotony preserving one. However, Lahtinen and Laurila (1990) demonstrated that the QO-spline, the monotony preserving taper curve formed by the QO-algorithm of Lahtinen (1988), may be even better than the MR-spline, the shape preserving taper curve formed by the algorithm of McAllister and Roulier. The main reason for the phenomen seems to be that the structure of the QO-algorithm is theoretically more advanced as explained by Lahtinen and Laurila (1990).

The QO-algorithm is in principle able to produce also shape preserving splines. In the investigation of Lahtinen (1988) it appeared, however, that shape preserving taper curves thus obtained did not give satisfactory volume estimates. The main obstacle seemed to be the parametrisation used in the first phase of the algorithm. Lahtinen (1990) has developed a new parametrisation for the QO-algorithm. This makes the preservation of the shape more natural and efficient. A detailed analysis of some interesting features is to be found in Lahtinen (1992).

Our intention is now to investigate whether the *QS-algorithm*, the new shape preserving form of the QO-algorithm, can produce a more reliable taper curve than the method of McAllister and Roulier or than the QO-algorithm. The theoretical differences of the methods are in favour of an affirmative result. For the investigation we formed taper curves with the shape preserving QS-algorithm using different sets of measured diameters from relative heights. The parameters of the quadratic splines are determined with the aim to obtain good volume estimates by the taper curve. The properties of the obtained *QS-splines* are examined by comparing them with the monotony preserving QO-splines and with the shape preserving MR-splines. The investigation is concluded with an analysis of some essential features of the QS-spline.

2 Materials and methods

2.1 Monotony preserving taper curves

A taper curve can be constructed in several ways by using spline functions as explained among others by Sloboda (1976), Lahtinen and Laasasenaho (1979), Lappi (1986), Lahtinen (1988), Lahtinen and Laurila (1990). We will call such taper curves *taper splines*. The method of Lahtinen (1988), called the QO-algorithm, uses quadratic splines. The aim of the method is to construct a taper curve which reproduces the measured diameters by interpolation and is in addition able to carry information about the shape of the stem.

The essential idea of the QO-algorithm is the use of a set of parameters in the presentation of the quadratic spline. The parameters determine directly or indirectly the first derivatives of the spline at interpolating points and the places of possibly needed additional breakpoints between interpolating points. The algorithm is local in the sense that an alteration of a parameter value changes the spline only at a certain neighbourhood of the point to which the parameter is attached. The implementation of the QO-algorithm used in Lahtinen (1988) is intended to preserve the monotony.

The QO-algorithm has essentially two phases. In the first one the parameters are determined by a preparatory method which at this early stage already takes some care on the wanted monotony or shape and the coefficients of the quadratic spline are evaluated. The form of the first phase presented in Lahtinen (1988) preserves monotony in normal cases. The analysis of the behaviour in exceptional cases is presented in Lahtinen (1992). In the second phase the preservation of the monotony or shape is examined. In the case of non-preservation the parameters causing the violation are altered according to the theory for a proper result. Because the method is local there is no alteration in the part of the spline already examined and accepted .

The essential aspects of the first phase of the QO-algorithm are in the original form as follows: Given initial data $D = (x_i, y_i)_1^n$ where the number y_i represents a measurement at point x_i it is defined auxiliary quantities

$$\delta_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, \qquad i = 1,\ldots,n-1$$

$$l_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}, \qquad i = 1,\ldots,n-1$$

$$\mu_i = \frac{l_{i-1}\delta_{i-1} + l_i\delta_i}{l_{i-1} + l_i}, \qquad i = 2,\ldots,n-1$$

After this the first derivative m_i of the quadratic spline at the point x_i is determined by means of parameter $\alpha_i \geq 0$,

$$m_1 = \delta_1 + \alpha_1(\delta_1 + \mu_2),$$

$$m_i = \begin{cases} \alpha_i\mu_i, & \text{if } \delta_i\delta_{i-1} > 0, \\ 0, & \text{otherwise} \end{cases} \qquad i = 2,\ldots,n-1$$

$$m_n = \delta_{n-1} + \alpha_n(\delta_{n-1} + \mu_{n-1})$$

On an interval $[x_i, x_{i+1}]$, where $m_i + m_{i+1} \neq 2\delta_i$, an additional breakpoint ξ_i is needed. It is determined by using a parameter $b_i \in [0, 1]$. In practice there is an additional breakpoint on almost every interval. For details see Lahtinen (1988).

The parameters α_i, $i = 1,\ldots,n$ and b_i, $i = 1,\ldots, n-1$ have to be determined within certain limits so that the resulting quadratic spline preserves monotony or shape as it was intended. These conditions do not determine the parameters uniquely. There is still some freedom which can be used to get the spline to fulfil additional requirements. In cases where the parameters α_i and b_i are defined by some iterative method the values $\alpha_i = 1$, $b_i = 0.5$ are reasonable initial values.

By using this parametrisation in a taper spline it is possible to get good volume and diameter estimates as shown in Lahtinen (1988). The derivative parametrisation has, however, some weaknesses.

At the end points the parametrisation of the derivatives does not always work properly. If in the data $y_1 = y_2 \neq y_3$, the procedure gives $m_1 \neq 0$, $m_2 \neq 0$, which does not give monotony on the interval $[x_1, x_2]$. Similarly in the case $y_{n-2} \neq y_{n-1} = y_n$ the procedure gives derivatives so that $m_{n-1} \neq 0$, $m_n \neq 0$ with the same consequences on the interval $[x_{n-1}, x_n]$.

This means that the first phase does not always preserve the monotony at end points. The defect is not very serious in the taper curve applications where it is very unusual to have the above kind of data. For instance in our test material of 1864 spruce stems there were only three trees with the above mentioned behaviour in the butt and none with this behaviour at the top.

Also at the inner points the derivative parametrization in the first phase may not always preserve the shape as explained in Lahtinen (1992).

A characteristic feature of this parametrisation is that the conditions for the preservation of the shape are quite complicated. This makes it difficult to construct an effective first phase which would take care of the preservation of the shape of a taper spline. In fact, the QO-algorithm in Lahtinen (1988) was implemented only for the preservation of monotony even though in many cases it also preserves the shape.

2.2 Shape preserving taper curves

The monotony preserving implementation of the QO-algorithm gives reliable taper splines when a sufficient number of diameter measurements is available at relative heights. Lahtinen and Laurila (1990) have shown that it may even surpass the use of the shape preserving quadratic spline of McAllister and Roulier (1981) as a taper spline.

In principle, however, a shape preserving taper curve should be superior to a monotony preserving one. The QO-algorithm can be implemented also to preserve the shape and it is theoretically more advanced than the algorithm of McAllister and Roulier. Therefore it seems worthwhile to investigate whether a more reliable shape preserving taper spline could be obtained in this way.

The preservation of shape is tied to the parametrisation of the QO-algorithm. As we remarked, the parametrisation is done in order to preserve monotony. In fact, it appeared during the investigation of Lahtinen (1988) that using the QO-algorithm it was in practice not always possible to have taper splines having both good volume estimates and the preservation of shape. The quality of volume estimates was then considered to be more important.

A more advanced parametrisation of the first phase of the QO-algorithm was presented in Lahtinen (1990). In it the derivative m_i at the interpolating point x_i was chosen by using a parameter a_i and the slopes δ_{i-1}, δ_i as follows:

$$m_1 = a_1\delta_1$$

$$m_i = \begin{cases} \delta_i + a_i(\delta_{i-1} - \delta_i), & \text{if } \delta_i\delta_{i-1} > 0, \\ 0, & \text{otherwise} \end{cases} \qquad i = 2,\ldots,n-1$$

$$m_n = a_n\delta_{n-1}$$

At an inner point x_i the parameter a_i is selected from a subinterval of [0,1] as explained in Lahtinen (1990). The value a_i = 0.5 can usually be used in the absence of additional information. Lahtinen (1992) has shown that this leads in certain cases asymptotically to the same result as the choice α_i = 1 in the older parametrisation. This asymptotic similarity especially takes place at the inner points of a taper spline.

At the end points of the interval the new parametrisation uses the initial information more completely than the old one. This means that the magnitudes of the parameters a_1, a_n are governed by the rules

$$\begin{array}{ll} 0 \le a_1 < 1 & \text{if } \delta_1(\delta_2 - \delta_1) > 0, \\ a_1 = 1 & \text{if } \delta_1(\delta_2 - \delta_1) = 0, \\ a_1 > 1 & \text{if } \delta_1(\delta_2 - \delta_1) < 0, \end{array}$$

and similarly

$$\begin{array}{ll} 0 \le a_n < 1 & \text{if } \delta_{n-1}(\delta_{n-2} - \delta_{n-1}) > 0, \\ a_n = 1 & \text{if } \delta_{n-1}(\delta_{n-2} - \delta_{n-1}) = 0, \\ a_n > 1 & \text{if } \delta_{n-1}(\delta_{n-2} - \delta_{n-1}) < 0, \end{array}$$

The treatment of the additional breakpoints is the same as in the older form of the QO-algorithm by Lahtinen (1988).

This parametrisation behaves better than the previous one. For instance in the situation where $y_1 = y_2 \neq y_3$ we have now $m_1 = m_2 = 0$ as it should be. Similarly the parametrisation is able to preserve the shape at inner points also in the cases where the earlier parametrisation did not. An important advantage of the new parametrisation is that it is much simpler to determine the parameter values which give a shape preserving spline. In many cases these values are independent of the data. This makes the preservation of the shape easier for a taper spline.

We will investigate the new implementation of the QO-algorithm as presented in Lahtinen (1990). In order to distinguish the two forms of the QO-algorithm we will call the algorithm *QS-algorithm* when the new parametrisation is used and reserve the name QO-algorithm to refer to the original parametrisation. Our aim is twofold, to examine the ability of the QS-algorithm to form taper curves and to estimate the quality of thus obtained shape preserving taper splines.

There is also in this new parametrisation some freedom in the parameters of a shape preserving quadratic spline. A standard method of the construction is to fix the remaining freedom of the parameters so that the taper spline has certain prescribed properties. Because of the difficulties mentioned earlier the crucial property is in the case of a taper spline the quality of the volume estimates. Therefore we will determine the parameters so that the resulting shape preserving taper spline reproduces as well as possible given proper volume estimates. The resulting taper spline will be called the *QS-spline*.

At the end points we have to choose the derivative parameters from three different intervals as explained earlier. In the case of a taper spline it is reasonable to expect that the case where $a_1 > 1$ and $a_n > 1$ would be dominant. Therefore we concentrated on determining the parameters for this case. In the case where $a_1 < 1$ we fixed the parameter value at the butt to be $a_1 = 0.99$ and similarly in the case where $a_n < 1$ we used a fixed value of $a_n = 0.99$ at the top.

2.3 The evaluation of the quality

The original version of the QO-algorithm could not always give taper splines with proper volume estimates and at the same time preserve the shape. Therefore we decided that the most consideration was whether the parameters of the shape preserving spline constructed by the QS-algorithm could be determined so that the corresponding taper spline, QS-spline, produced proper volume estimates. As proper volume estimates we considered in this investigation the ones given by the QO-spline QO15 presented in Lahtinen (1988) which passes through fourteen measured diameters.

After the binding of parameters on the basis of volume estimates the taper spline is completely determined and its quality can be estimated. This can take place on the basis of the volume, diameter values and the form produced by the taper spline in a representative sample tree material. In practice it is done by comparing the properties of the taper spline with the ones of another taper curve.

In this investigation we compared the new shape preserving QS-spline both to the monotony preserving QO-spline and to the shape preserving MR-spline. Because the relation of QO-spline and MR-spline has been investigated in Lahtinen and Laurila (1990) we will only present here numerical comparisons with the MR-spline in the cases which cannot be derived from the earlier investigation.

Another important thing is the amount and placement of measurements used in the construction of a taper curve. It is of course advantagenous to have a reliable taper curve which can be formed with only few initial measurements. On the other hand it is obvious that the less information used in its construction the coarser the taper curve. We wanted to have some idea how much initial information is needed for a reliable QS-spline.

This was investigated by first constructing a basic QS-spline passing through several measured diameters and examining after that how much the quality of a QS-spline is weakened when the number of initial measurements is gradually reduced.

As sample tree material where the determination of parameters and the comparisons were made we used the same 1864 spruce stems as Lahtinen (1988). A detailed description of the material is to be found in Laasasenaho (1982). We will only mention the essentials here. For each tree the tree height was recorded and diameters were measured at 14 relative heights from the ground namely 1, 2.5, 5, 7.5, 10, 15, 20, 30, 40, 50, 60, 70, 80 and 90 %. The diameter at the top was always taken to be 0.4 cm.

In order to estimate the properties of the QS-spline we evaluated in the sample tree material the mean relative volume differences and mean maximal absolute values of diameter differences as well as their standard deviations by diameter classes. These differences were calculated in relation to the comparison taper curve for the whole stem and for seven consecutive parts into which the stem was divided. The limits of these parts were at relative heights of 1, 5, 10, 20, 40, 60, 80 and 100 %. We also evaluated errors of taper curves at the heights where the measured diameters were not used in the construction and tabulated the mean errors and standard deviations by diameter classes.

One important criterion is the form of the taper curve. This is needed for instance in dividing a stem into assortments of lumber. It is not easy to find any analytical way to measure the form. Therefore we chose representatives of the most typical forms among the sample trees. The graph of the tested taper curve and the one of the comparison taper curve were drawn for these trees.

For the evaluation of the effect of the initial information we used several different sets of measurements as interpolating points. The trend of the conclusions can be presented by the results at three different point sets used already in Lahtinen (1988) with the notation introduced there. In this mentioned investigation it appeared that the placement of measurements in these three sets is quite reasonable for a good taper spline. The results in De Boor (1978) show that even better choices are possible, however this does not apply to our situation where the measurements have already been done.

The first set of measurements consists of all the 14 diameters available together with the fixed top diameter. This is called simply *set 15*. The best shape preserving taper spline constructed by the QS-algorithm and interpolating at set 15 is denoted by QS15 and called the QS-spline.

The second set consists of 7 measured diameters for the heights 1, 5, 10, 20, 40, 60, 80 % and the top diameter. The best shape preserving taper spline constructed by the QS-algorithm and interpolating at this set 8A is denoted by QS8A and also called the QS-spline. The third set consists of three measured diameters for the heights 2.5, 10, 50 % and the top diameter. The best shape preserving taper spline constructed by the QS-algorithm and interpolating at this set 4B is denoted by QS4B and again called the QS-spline.

The comparison taper splines used the same sets of measurements as the interpolating points. We used the monotony preserving QO-splines of Lahtinen (1988), namely the taper splines QO15, QO8A and QO4B for their good properties. For reference we also used the shape preserving MR-splines of McAllister and Roulier presented in Lahtinen and Laurila (1990). These were MR15 and MR8A. We constructed for comparison also a MR-spline for the point set 4B and denoted it as MR4B.

3 Results

3.1 Taper curves through several measured diamenters

The first task in the investigation was to examine whether the QS-algorithm could, unlike the QO-algorithm, produce shape preserving taper splines with proper volume estimates. This was tested with a shape preserving taper spline using all the measured diameters available. Using the QS-algorithm we constructed a shape preserving quadratic spline interpolating at set 15. The parameters of the quadratic spline were determined by interactive iteration so that the seven mean partial volumes in the sample tree material given by this taper spline differed from the corresponding values given by the monotony preserving QO-spline QO15 as little as possible.

The obtained parameter values of the resulting QS-spline QS15 are in Table 1. There x_i is the relative height where the diameter has been measured, a_i is the parameter of the first derivative of the taper spline at x_i and b_i is the parameter of the possible additional breakpoint on the interval $]x_i, x_{i+1}[$. The distribution of end point derivative parameters is given in Table 2. As was expected almost all the trees belong to the group where the derivative parameter at the butt, $a_1 > 1$. On the other end, on the top only two-thirds of trees have the expected behaviour, that is have $a_n > 1$. Some of the comparisons of the taper splines QS15 and QO15 are presented in Tables 3–4.

The determination of parameters succeeded so well that the shape preserving taper spline QS15 gives in the whole sample tree material the same mean total and partial volumes as the monotony preserving taper spline QO15. As Lahtinen and Laurila (1990) remarked, the taper splines constructed with the QO-algorithm were not always able to preserve the shape while giving good volume estimates. This shows clearly that the new QS-algorithm works in shape preserving taper splines better than the QO-algorithm. Also other indicators show that the taper splines QS15 and QO15 differ very little. The standard deviations of the relative volume differences are small, 0.07 on the whole stem, and the total volume difference is less than 0.2 % for almost every sample tree, that is for 99.25 % of trees. The mean maximal absolute value of diameter differences is also small, being at the butt less than 0.3 cm and above it less than 0.1 cm.

The volume comparisons of QS15 and QO15

Table 1. Parameter values of taper spline QS15.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x_i	1	2.5	5	7.5	10	15	20	30	40	50	60	70	80	90	100
a_i	2.8	0.24	0.5	0.45	0.52	0.32	0.33	0.54	0.5	0.5	0.5	0.5	0.5	0.5	1.1
b_i	0.737	0.5	0.5	0.5	0.3	0.2	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	

Note: a_1 and a_{15} are at certain cases chosen differently, see Table 2.

Table 2. The relative (%) frequency of different choices of derivative parameters at the butt and top.

Taper spline	Butt $a_1 > 1$	$a_1 = 1$	$a_1 = 0.99$	Top $a_n > 1$	$a_n = 1$	$a_n = 0.99$
QS15	95.4	0.2	4.4	65.1	6.8	28.1
QS8A	99.5	0.0	0.5	78.5	2.9	18.6
QS4B	97.5	0.1	2.4	95.5	0.0	4.5

by diameter classes show some mean total volume differences that would be expected from the standard deviations. They have a certain tendency. For small trees QS15 seems to give smaller volumes and for large trees larger volumes than QO15. These differences are however small, less than 0.05 % in absolute value. For small trees the smaller volumes of QS15 are due to the behaviour at the butt. The smaller butt volumes may be nearer the real values as explained in Lahtinen and Laurila (1990). In diameter classes 9–23 cm both taper splines give the same mean total volume.

The standard deviations of the total volume differences vary in diameter classes between 0.02 and 0.14 without any clear tendency. When the partial volumes are examined by diameter classes it is noticed that the taper splines QS15 and QO15 differ in the interval [40,60] only for smallest trees and in the interval [60,80] not at all. This is not surprising because the parameters of the QO15 presented in Lahtinen (1988) give the impression that QO15 also preserves the shape on these intervals. As an approximative rule we can namely say that in the older form of the algorithm the shape is usually preserved around an inner point x_i where the parameter $\alpha_i = 1$. The more this parameter differs from the value 1 the more probable it is that the shape is not preserved in the neighbourhood of x_i.

The graphical investigation confirms that the taper splines QS15 and QO15 are very similar. They differ usually only at the places where the latter curve is not a shape preserving one. Even in these places the differences are fairly small as has been confirmed by the mean maximal diameter differences, too. So the graphs of QS15 and QO15 in the figures seem to coincide most of the time.

The results seem to imply that in practice QS15 is at least as good as QO15, and for small trees it may even be slightly better. By Lahtinen and Laurila (1990) this indicates that QS15 is at least as good as MR15. QS15 seems to be superior to MR15 at the butt; on other parts of the stem the differences are very small.

The good quality of the QS-spline QS15 shows that the QS-algorithm is as a shape preserving algorithm clearly better than the QO-algorithm.

3.2 Taper curves through seven measured diameters

After the investigation of the ability of the QS-algorithm to form proper shape preserving taper curves we turned our attention to the dependence of the obtained shape preserving taper curve, QS-spline, from the initial information. For this purpose we formed QS-splines with several measured diameters, but essentially less than in the previous case. From the many taper splines we tested we only present one here. We believe that this is sufficient to convey the general features of this case.

In alternative presented we used only half of the measurements. The chosen set 8A has seven measured diameters and the fixed top diameter. Compared to set 15 every second measurement was left out. The QS-algorithm was used to construct a shape preserving quadratic spline interpolating in set 8A. Now the criterion for the determination of the parameters of the spline was that in the sample tree material the seven mean partial volumes given by this taper spline should differ as little as possible from the corresponding values of the taper spline QS15. The parameter values of the resulting QS-spline QS8A are presented in Table 5 where the notation is the same as in Table 1. The distribution of derivative parameters at the end points is given in Table 2. At both end points the number of trees with the expected parameter behaviour has increased but is at the top still only 78.5 %. Some of the properties of taper spline QS8A are presented in Tables 3, 4 and 6.

The results show that taper spline QS8A does not differ much from taper spline QS15. From the seven mean partial volumes we evaluated only the top volumes vere different. More exactly, on the interval [80, 100] the taper spline QS8A gives 1.16 % smaller mean volume than QS15. This does not have much practical significance except that it causes a –0.08 % difference to the mean total volumes. The standard deviations show that although the mean partial volumes coincide there are differences on single trees even if they are not large. For instance, the

Table 3. Mean relative (%) total and partial volume differences and their standard deviations.

Compared taper splines	Mean volume differences								Standard deviations							
	st-100	st-5	5–10	10–20	20–40	40–60	60–80	80–100	st-100	st-5	5–10	10–20	20–40	40–60	60–80	80–100
QS15–QO15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.07	0.36	0.10	0.22	0.10	0.03	0.00	0.30
QS8A–QS15	–0.08	0.00	0.00	0.00	0.00	0.00	0.00	–1.16	1.17	4.63	2.05	2.20	2.28	2.59	4.05	10.79
QS8A–QO8A	–0.06	–0.32	–0.03	0.06	–0.03	–0.04	–0.08	–1.33	0.30	0.62	0.10	0.67	0.74	0.21	0.63	2.27
QS4B–QS15	–0.19	–0.04	0.13	–0.03	–0.03	–0.04	0.01	6.27	2.62	2.67	3.27	3.72	5.14	2.18	12.71	33.31
QS4B–QO4B	0.61	–0.42	0.05	0.49	–0.37	0.23	5.01	3.28	1.70	6.07	0.32	1.08	2.26	0.97	8.29	18.17
QS4B–MR4B	1.76	–0.46	–2.60	2.94	5.19	0.37	–1.42	3.04	0.79	3.14	1.18	1.22	1.90	0.20	3.28	7.22

st = stump; percentages are always calculated with regard to the corresponding volumes given by the latter taper curve.

Table 4. Averages of maximal absolute values of diameter differences (cm) and their standard deviations.

Compared taper splines	Averages of maximal absolute values								Standard deviations							
	st-100	st-5	5–10	10–20	20–40	40–60	60–80	80–100	st-100	st-5	5–10	10–20	20–40	40–60	60–80	80–100
QS15–QO15	0.31	0.29	0.04	0.06	0.02	0.00	0.00	0.03	1.08	1.08	0.03	0.06	0.02	0.01	0.00	0.03
QS8A–QS15	0.85	0.75	0.24	0.23	0.23	0.21	0.23	0.32	0.64	0.65	0.24	0.26	0.22	0.19	0.21	0.31
QS8A–QO8A	0.57	0.56	0.03	0.08	0.07	0.02	0.04	0.14	0.49	0.50	0.02	0.08	0.09	0.02	0.04	0.09
QS4B–QS15	1.46	1.34	0.45	0.42	0.50	0.39	0.58	0.58	1.27	1.31	0.39	0.35	0.35	0.28	0.41	0.44
QS4B–QO4B	0.73	0.50	0.06	0.11	0.27	0.32	0.51	0.48	2.08	2.09	0.03	0.12	0.16	0.17	0.30	0.30
QS4B–MR4B	2.26	2.26	0.54	0.46	0.54	0.19	0.16	0.18	1.60	1.61	0.36	0.27	0.30	0.11	0.12	0.13

st = stump

Table 5. Parameter values of taper spline QS8A.

i	1	2	3	4	5	6	7	8
x_i	1	5	10	20	40	60	80	100
a_i	3.31	0.28	0.24	0.99999	0.45	0.38	0.21	1.62
b_i	0.7	0.5	0.64	0.09	0.5	0.55	0.01	

Note: a_1 and a_8 are at certain cases chosen differently, see Table 2.

total volumes produced by QS8A and QS15 differ less than 1 % in two-thirds of the sample trees.

The volume comparisons of QS8A and QS15 by diameter classes confirm that there are small differences in the mean volumes of diameter classes. Unlike the comparison of QS15 and QO15 there is no apparent tendency here. The greatest relative differences take place in the smallest trees with 1 cm breast height diameters.

The similarity of taper splines QS8A and QS15 can also be seen from the mean maximal absolute diameter difference. It is at the butt less than 0.8 cm, between the butt and the top less than 0.3 cm and on the top slightly over 0.3 cm. These are quite small numbers. However, the comparison reveals that the greatest absolute difference from the butt upwards is on the top. This means that the otherwise good ability of QS8A to simulate QS15 has not quite succeeded on the top.

Because all the measurements were not used in the construction of the taper spline QS8A we can evaluate the diameter errors of QS8A in the remaining measurement points. The mean values and standard deviations are in Table 6. The greatest mean error, –0.1 cm is at the 90 % height. While not being large it however, together with the mean maximal diameter difference, reveals that this taper spline runs into some difficulties at the top. At other points of the stem the mean absolute value of the error is less than 0.05 cm which is practically negligible. The standard deviations imply that most diameter errors are near the mean errors. The results of these considerations show that the quality of the taper spline QS8A is very good.

When the QS-spline is compared with the MR-spline as presented in Lahtinen and Laurila (1990) it can be noticed that the taper spline MR8A gives too large volumes and diameters at the butt and slightly too small volumes and diameters on the interval [20, 40]. On both areas QS8A behaves very correctly. MR8A is better only at the top. On other parts of the stem the differences are very small. On the whole this indicates that QS8A should be preferred to MR8A.

The differences between QS- and QO-splines are still small in the case of seven measured diameters as seen from Tables 3, 4, 6 and the results of Lahtinen (1988). Some discrepancies begin, however, to appear indicating that for a taper spline the preservation of shape may be for this number of initial measurements better than the preservation of monotony. Especially the QS-spline QS8A seems to have a slightly better behaviour at the butt than the QO-spline QO8A.

The graphical investigation gives support for the above mentioned features. The graphs of QS8A and QS15 seem to coincide in most figures. Differences take usually place for trees with irregularities which can not be seen from the smaller set of measurements. The differences of the taper splines QS8A and QO8A seem in general to take place only at the parts of the stem where the QO-spline does not preserve the shape.

In conclusion we can say that the differences between the treated taper splines interpolating in the set 8A are in general small. However, the shape preserving taper spline QS8A can be considered to be better than the other shape pereserving taper spline MR8A and to be at least as good as the monotony preserving taper spline QO8A. For many purposes QS8A is about as good as the taper spline QS15 using twice as many measured diameters.

3.3 Taper curve through three measured diameters

The smallest amount of initial information we used in the tests was three measured diameters and the height. The point set 4B offers a suitable case for presenting the properties of a shape preserving QS-spline with few initial measurements. The characteristic feature of the set 4B is

Table 6. The mean diameter errors (cm) with regard to the measured diameters, and their standard deviations.

Taper spline	me(d) st.d.	Relative (%) height 1	2.5	5	7.5	10	15	20	30	40	50	60	70	80	90
QS8A	me(d)		–0.04		0.03		–0.01		–0.01		–0.01		0.00		–0.10
QO8A	me(d)		0.09		0.04		0.00		–0.01		0.00		0.00		–0.04
MR8A	me(d)		0.59		0.04		0.01		–0.06		–0.01		0.00		–0.05
QS4B	me(d)	0.43		–0.01	0.02		0.00	–0.05	–0.01	0.04		–0.08	–0.08	–0.06	–0.05
QO4B	me(d)	0.68		–0.05	0.03		–0.05	–0.12	0.01	0.19		–0.36	–0.32	–0.24	–0.12
MR4B	me(d)	–2.59		0.53	0.25		–0.29	–0.51	–0.48	–0.15		0.01	0.00	–0.07	–0.13
QS8A	st.d.		0.84		0.33		0.34		0.31		0.28		0.30		0.42
QO8A	st.d.		0.86		0.33		0.31		0.31		0.28		0.30		0.42
MR8A	st.d.		0.92		0.33		0.32		0.31		0.28		0.30		0.42
QS4B	st.d.	2.38		0.52	0.39		0.43	0.46	0.44	0.36		0.39	0.53	0.61	0.59
QO4B	st.d.	2.71		0.52	0.39		0.43	0.48	0.47	0.42		0.45	0.72	0.85	0.68
MR4B	st.d.	2.24		0.60	0.41		0.47	0.56	0.52	0.36		0.39	0.53	0.61	0.59

me(d) = mean diameter error (cm) at a relative height
st.d. = corresponding standard deviation

that the lowest interpolating point has been raised to the 2.5 % height in order to obtain better behaviour at the butt. This means that the taper curve on the interval [stump, 2.5] must be formed by extrapolation which does not in general give as stable behaviour as interpolation.

The shape preserving QS-spline was again constructed with the QS-algorithm. The parameters of the taper spline were determined as before by interactive iteration so that the seven mean partial volumes differed as little as possible from the corresponding volumes given by the taper spline QS15. The parameter values of the resulting QS-spline QS4B are in Table 7 where the notation is the same as in Table 1. The distribution of derivative parameters at end points is presented in Table 2. Now almost all the trees are in the expected groups. Some of the properties of QS4B are given in Tables 3, 4, 6, 8 and 9.

The determination of parameters succeeded quite well apart from at the top even if the partial volume differences could not be pushed to zero with the same accuracy as in the case of several measured diameters. The remaining differences produced a –0.19 % mean total volume difference between taper splines QS4B and QS15 in the sample tree material. On the sensitive extrapolation area on the butt the difference was only –0.04 %. This indicates that the extrapolation has been succesful which is in accordance with

Table 7. Parameter values of taper spline QS4B.

i	1	2	3	4
x_i	2.5	10	50	100
a_i	4.16	0.0001	0.52	1.36
b_i	0.5	0.418	0.45	

Note: a_1 and a_4 are at certain cases chosen differently, see Table 2.

Table 8. Mean relative (%) total and partial volume differences of QS4B and QS15 by diameter classes.

Diameter (cm)	Interval st-100	st-5	5–10	10–20	20–40	40–60	60–80	80–100
1	–0.07	–0.76	–0.65	0.46	–0.92	–2.42	7.99	28.88
3	1.46	–1.26	0.36	0.52	0.57	0.62	12.34	30.72
5	0.11	–2.20	–0.20	–0.44	–0.30	0.02	5.82	26.23
7	0.13	–1.00	0.38	0.38	–0.07	0.18	3.70	17.30
9	0.24	0.26	0.27	0.39	0.30	–0.13	1.83	16.19
11	–0.05	–0.02	0.09	–0.20	–0.15	–0.13	1.87	18.64
13	0.29	0.13	0.04	0.53	0.93	0.18	0.52	10.96
15	–0.26	–0.10	0.18	–0.04	–0.25	0.00	0.23	4.07
17	–0.37	0.09	–0.42	–0.34	0.18	0.07	–1.31	0.87
19	–0.13	0.41	–0.14	0.23	–0.15	–0.10	–0.07	5.36
21	–0.55	0.10	0.26	–0.51	–0.34	–0.04	–1.67	–3.10
23	–0.43	0.27	0.22	–0.38	0.10	0.08	–2.16	–0.79
25	–0.50	0.34	0.40	–0.39	–0.54	0.04	–1.58	–0.51
27	–0.56	0.28	–0.07	–0.23	–0.25	–0.25	–2.23	–0.61
29	–0.48	0.08	0.38	0.51	–0.01	–0.19	–4.31	–5.29
31	–0.55	0.04	0.46	–0.21	–0.38	–0.39	–1.79	1.03
33	–0.70	0.14	0.84	–0.41	–1.09	–0.20	–1.78	–3.66
35	0.11	0.06	0.15	1.22	1.74	–0.10	–5.30	–8.32
37	–0.45	0.18	–0.17	0.16	0.47	0.01	–4.16	–9.54
39	–1.20	0.13	–0.29	–1.24	–1.72	–0.42	–1.54	–4.18
41	–2.02	0.70	3.14	–1.98	–3.43	–1.70	–5.44	1.91
43	0.84	0.29	1.90	–0.08	1.88	0.47	2.07	–7.53
45	–1.37	0.01	0.75	–1.03	–1.95	–1.14	–4.32	–7.12
47	0.03	–1.24	–0.21	1.37	–1.75	–0.82	5.78	21.50
53	0.18	–0.81	–3.31	3.92	2.88	–1.20	–6.82	–17.77
61	0.38	0.30	1.62	0.92	2.35	–0.89	–2.61	–15.22
mean	–0.19	–0.04	0.13	–0.03	–0.03	–0.04	0.01	6.27

st = stump

Table 9. Standard deviations of relative (%) total and partial volume differences of QS4B and QS15 by diameter classes.

Diameter (cm)	st-100	st-5	5–10	10–20	20–40	40–60	60–80	80–100
	Interval							
1	4.91	1.71	6.70	9.34	8.37	1.69	10.39	91.24
3	5.31	5.88	5.12	4.85	8.06	4.89	25.49	45.04
5	3.59	4.98	4.37	4.27	6.55	3.01	18.88	48.07
7	3.23	3.03	4.58	4.87	6.91	2.90	17.20	37.75
9	2.37	2.06	3.22	4.31	5.43	2.06	13.06	35.94
11	2.57	2.26	3.34	3.33	4.90	2.09	14.88	41.61
13	2.77	2.46	3.46	4.09	5.72	2.01	13.36	37.30
15	2.89	2.23	2.96	4.49	5.48	1.92	11.54	31.56
17	2.43	2.19	3.26	2.69	4.62	2.42	10.25	25.47
19	2.63	3.29	3.14	4.25	5.21	1.94	10.97	28.34
21	2.21	2.14	3.32	3.38	4.33	1.88	9.80	23.02
23	2.07	2.01	2.55	2.94	4.54	1.86	9.41	29.95
25	2.14	1.69	2.63	2.85	4.30	1.87	9.11	22.52
27	2.17	1.80	2.23	3.19	4.55	1.92	10.82	26.50
29	1.63	2.07	2.68	2.59	3.66	1.66	7.52	15.35
31	2.13	1.98	2.76	3.01	4.73	1.90	10.10	31.34
33	2.19	2.59	3.06	3.08	4.21	2.17	9.39	19.00
35	1.96	2.28	2.74	3.88	3.63	1.87	8.11	22.47
37	1.57	3.61	2.57	3.14	2.94	1.38	7.32	16.27
39	1.55	3.32	2.74	2.73	3.04	1.59	8.12	15.97
41	2.30	1.56	2.02	3.04	5.52	1.33	10.30	39.96
43	1.65	1.93	0.78	3.41	0.63	2.03	15.62	19.04
45	2.25	2.14	1.14	2.45	4.62	2.17	7.31	13.94
47	4.87	3.17	2.37	4.22	7.11	1.79	6.11	43.63
53	2.68	6.88	8.45	3.87	0.41	1.94	8.95	32.45
61	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
mean	2.62	2.67	3.27	3.72	5.14	2.18	12.71	33.31

st = stump

the earlier results of Lahtinen (1988). On almost the whole stem, that is on the parts of the interval [10, 80], the mean volume difference is less than –0.05 %.

In general the QS-spline QS4B tends to give smaller volumes than QS15. As can be seen from Tables 8 and 9, this behaviour is seen in almost all diameter classes, only the largest trees form an exception. Volume differences evaluated by diameter classes together with standard deviations seem also to imply that there are larger relative differences in the volumes of very small trees and very large trees than elsewhere.

The weakest part of the taper spline QS4B is the top where there are the greatest relative volume differences with large standard deviations. Also other QS-splines have had difficulties at the top but not as much as in this case. This can partly be explained by the small number of initial measurements. The top derivative of the spline has to be determined so that we have both the preservation of the shape and decent volume estimates on the parts of the long interval [50, 100]. This approach seems not to be able to cope with the behaviour near the top.

One indicator of the difference between QS-splines QS4B and QS15 is the mean maximal absolute diameter difference. It is on the extrapolation area, on the butt, 1.34 cm and on the rest of the stem less than 0.6 cm. This together with a tolerable standard deviation indicates that these taper splines do not differ much on regular trees. It can also be seen that the extrapolation produces large differences for some trees, but this is not a common phenomenon.

The mean diameter errors at the points where the measured diameters are not used in interpolation are fairly small; at the butt 0.43 cm and above it less than 0.1 cm. The standard deviations are small except on the extrapolation area at the butt.

The graphical investigation agrees with the

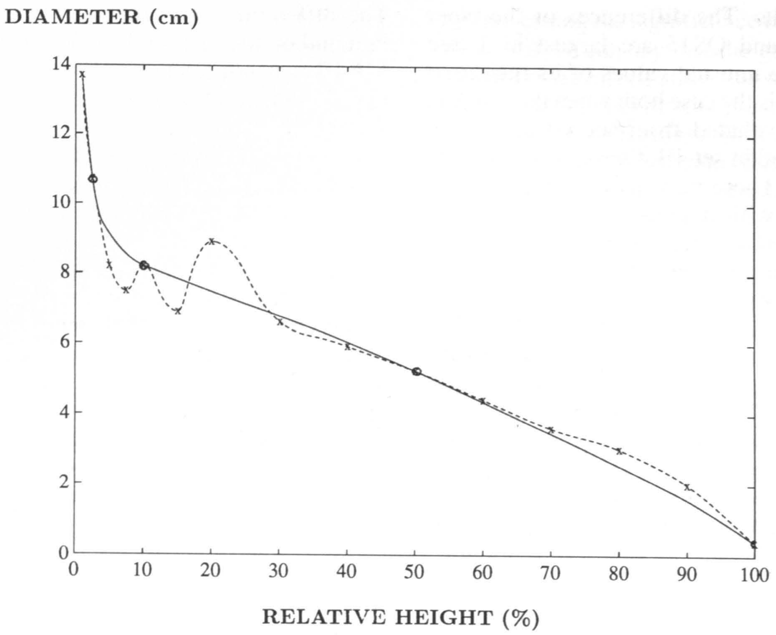

Fig. 1. Taper curves QS4B (connected line) and QS15 (dotted line) for an irregular tree (○ stands for an interpolating point of QS4B and × stands for an interpolating point of QS15).

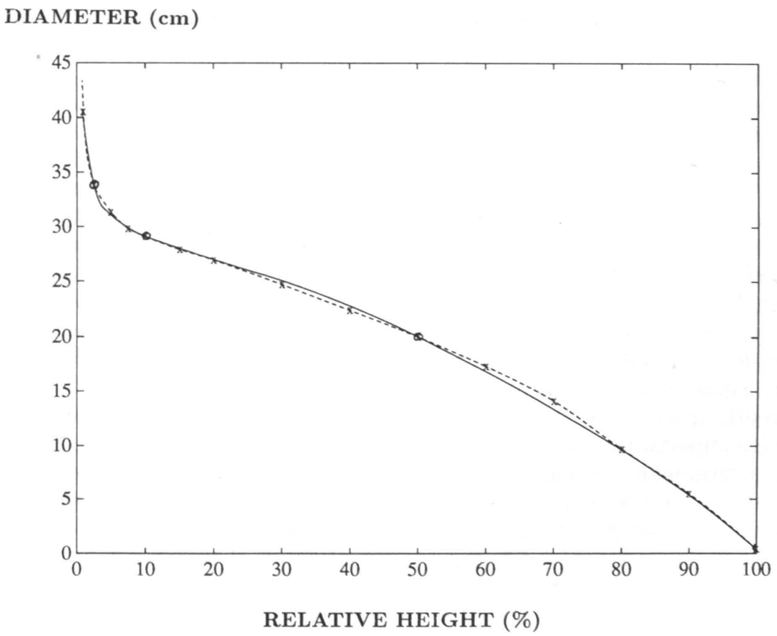

Fig. 2. Taper curves QS4B (connected line) and QS15 (dotted line) for a regular tree (○ stands for an interpolating point of QS4B and × stands for an interpolating point of QS15).

numerical results. The differences of the taper splines QS4B and QS15 are largest in a tree which has some unusual values of its measured diameters. This is the case both when these measurements are excluded from set 4B and when they are included in set 4B. One example can be seen in Fig. 1. There may also be small differences in trees with a regular set of measurements. A typical case is presented in Fig. 2. These two figures demonstrate also that the QS-spline QS4B as a rule gives a natural form to the tree.

The comparisons with other taper splines interpolating at the set 4B are plainly in favour of the QS-spline. This can most clearly be seen from the mean diameter errors presented with their standard deviations in Table 6. The taper spline QS4B is distinctly superior to the other shape preserving taper spline, MR-spline MR4B. The differences are especially apparent on the butt and on the interval [20, 40]. The MR-spline MR4B was not treated in Lahtinen and Laurila (1990) and therefore it is included in Tables 3, 4 and 6.

The monotony preserving QO-spline is better than the MR-spline but the comparisons presented in Tables 3, 4 and 6 imply that the QS-spline QS4B is superior to the QO-spline QO4B, too. In fact, from the results of Lahtinen (1988) it can be seen that QS4B is approximately as accurate as the QO-spline QO5C which is based on four measured diameters, which is one more than is used for QS4B.

As a whole the QS-spline QS4B can be considered as a satisfactory taper curve. In fact, it seems to be surprisingly good in relation to the small amount of initial information.

4 Discussion

4.1 The basis of the investigation

In principle a shape preserving spline function has better qualifications for a taper curve than a spline function preserving only the monotony. The investigation of Lahtinen and Laurila (1990) indicated that this theoretical advantage is not necessarily realisable in practice. In the comparison the monotony preserving QO-spline by Lahtinen (1988) seemed to be slightly superior to the shape preserving MR-spline of McAllister and Roulier (1981).

Some explanation for this phenomenon can be deduced from Lahtinen (1990) where it is shown that the demand for preservation of the monotony or the shape does not uniquely determine the spline. There is still freedom in the choice of the coefficients of the spline. This means that it is also important to have a satisfactory way for the determination of the spline coefficients within the general demand for preservation of the monotony or shape. The QO-algorithm appears to be in this sense more advanced than the shape preserving algorithm of McAllister and Roulier even if the QO-algorithm preserves only the monotony.

In fact, the QO-algorithm is also able to produce a shape preserving spline. In the investigation of Lahtinen (1988) it appeared, however, that the taper curve, QO-spline, produced by the QO-algorithm was not always able to have both good volume estimates and the preservation of the shape. In this situation the quality of volumes estimates was considered to be more important.

An analysis of the situation revealed that the parametrisation of the QO-algorithm was such that the preservation of the shape was in theory possible but in practice difficult. Lahtinen (1990) succeeded in developing a new implementation of the QO-algorithm called the QS-algorithm. In this the preservation of the shape could be done in a more efficient and natural way.

The aim of the present investigation was to examine the ability of the QS-algorithm to produce shape preserving taper curves and to estimate the quality of thus obtained taper splines. The starting point was the determination of the parameters of the shape preserving quadratic spline so that the resulting taper curve could give as accurate volume estimates as possible. In this investigation it was assumed that this could be achieved if the volumes were essentially the same as the ones given by the QO-spline QO15 through fourteen measured diameters.

After the determination of the taper spline by volume estimates the quality of the obtained QS-spline was estimated using the volume and diameter estimates and the form of the taper spline. For this purpose the QS-spline was compared to the earlier QO- and MR-splines.

One factor in the quality is the number of intial measurements needed for a sufficiently good accuracy in the taper curve. It is to be expected that a shape preserving taper spline would need less measurements than other taper splines. This was investigated by gradually reducing the number of measured diameters used in the construction of the QS-spline and comparing the result both with the QS-splines using more measurements and with QO- and MR-splines using the same measurements. It must, however, keep in mind that our method is based on the use of several measurements. When only very little information is available, then more statistical arguments must be used.

4.2 The quality of the QS-spline

Our aim was to develop a better shape preserving taper curve than the MR-spline. In this we have succeeded, the QS-spline seemed in our tests to be superior to MR-spline, although in many ways they behaved similarly. This is due to the common property of shape preserving. Even if the differences were in many cases small they were almost always in favour of the QS-spline. The only notable exception was that at the top part of the stem, that is from the relative height 80 % upwards, the MR-spline was sometimes better.

The essential question was whether the shape preserving taper spline constructed with the QS-algorithm was of the same quality as the QO-spline. The comparisons of taper splines in the investigation give an affirmative answer to this question. The significance of the observed differences depends of course on the nature of the other taper curve and on the number of the initial measurements.

The QS-spline is able to give similar volume estimates as the monotony preserving QO-spline while still preserving the shape. This is a very positive result. Generally speaking, the QS-spline tends in almost all aspects to be superior to the QO-spline even if the differences are in many cases small. The situation is to be expected because both taper splines are based to the same mathematical theory behind the different implementations of the QO-algorithm. This means that in some cases the QO-spline preserves in addition of the monotony also the shape.

Concerning the significance of the initial measurements we can say that the greater the amount of the initial information the smaller the differences in taper curves both preserving the monotony should be. Because both the QS-and QO-splines are fairly accurate, differences between them cannot be large when a large set of initial measurements is available.

When several measured diameters are available the shape preserving QS-spline is superior to the shape preserving MR-spline on the butt and at least as good elsewhere. With the same initial information the QS-spline is able to produce as reliable total and partial volume estimates as the monotony preserving QO-spline. The measurable diameter errors were quite small for the QO-spline, however, in the QS-spline they are even smaller.

When only a small amount of the initial information is available then the differences are more clear. The QS-spline tends to be superior to the QO-spline both with regard to volume and diameter estimates and with the form of the taper curve. Especially noteworthy is that the QS-spline does not have the same problems with dealing with small trees that is observed with the QO-spline. The MR-spline seems to be inferior to both these splines.

As a whole the results indicate that the shape preserving QS-spline can be considered a better taper curve than the monotony preserving QO-spline. It can also be considered to be better than the other shape preserving taper spline, MR-spline. The differences are the larger the smaller the number of measured diameters in the construction.

4.3 Some features of the QS-spline

The results confirm that the QS-spline is a very reliable taper curve when suitable intitial information is available. However, quality indicators together with the interactive iteration used for the determination of the parameters reveal that the QS-spline has also some properties that must be treated with care.

It was known beforehand that as a quadratic spline it has difficulties in adapting itself to the fast tapering of the butt. The QS-spline is, however, no worse than the other tested taper splines based on quadratic splines, namely the QO-spline and MR-spline. A new feature was that the QS-spline has difficulties also at the other end of the stem, at the top. The standard deviations of different indicators are large and relative diameter errors greater than in other parts of the stem. The mean absolute diameter errors at the top are on the other hand only about 0.1 cm. As the practical significance of the top is small this weakness seems to be tolerable.

The shape preserving MR-spline investigated by Lahtinen and Laurila (1990) had difficulties in giving accurate volume or diameter estimates between the relative heights of 20 % and 40 %. Although the QS-spline tends to be superior to the MR-spline it seems to have the same kind of difficulties.

For instance in the case of seven measured initial diameters it was very tedious to determine the parameters so that the volume estimates for the interval [20, 40] were within the prescribed tolerances. In fact the parameter values had to be pushed into extereme acceptable positions leaving no room for further alterations. Also other sets of initial measurements produced similar situations. It seems to be an intrinsic property of a shape preserving taper spline that it tends to give in the lower part of the stem smaller volumes and diameters than other taper splines including the cubic spline of Lahtinen and Laasasenaho (1979).

Concerning the similarity of the shape preserving QS-spline and the monotony preserving QO-spline it has also to be remembered that the measurements are not always in accordance to the standard form of the tree. A normal tree is convex near the stump, has one point of inflection and above this it is concave. If the form of a set of measurements is defined to be the form of a linear spline function interpolating at this set we can investigate the form of the measurements. It appears that the set of measurements of a tree has as a rule several points of inflection. This phenomenon gives some reason to consider the significance of the preservation of the shape when a large set of measurements is available.

The situation is totally different when only a few diameters have been measured from a tree. Then the preservation of shape tends to give the taper curve the natural form of a tree which would otherwise be very difficult to obtain. This natural form definitely makes the errors of the taper curve smaller both in volume and diameter estimates.

This shape preserving property can be utilised in a situation where the level of accuracy of the estimates is determined before the measurements are made. Then the shape preserving taper spline is easier and more economical to use because it needs less initial information than other taper splines for a given accuracy. This makes the QS-spline superior to the QO-spline which in its turn is superior to a common interpolating taper spline. This is the more apparent the smaller number of initial measurements is in question.

The remaining freedom in the choice of parameters in a shape preserving taper spline works on two ways. When some additional information is available then it is an advantage because it can be used to adapt the taper spline to the information which increases the accuracy. On the other hand, if no additional information is availabe then we still have to decide how to determine the parameters in the taper spline. This results in a level of uncertainty which has to be resolved somehow. In our investigations we have determined the parameters by volume estimates in a representative sample tree material. Because the diameters have been measured at relative heights we have been able to utilize in the determination the assumption that the form of the stem is independent of the size of the tree.

In conclusion we can state that the theoretical properties of a shape preserving quadratic spline raise expectations of the quality of taper splines. The shape preserving QS-spline fulfils these expectations quite satisfactorily.

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