A Comparison of Two Parameter Prediction Methods for Stand Structure in Finland

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The objective of this paper was to predict a model for describing stand structure of tree heights (h) and diameters at breast height (dbh). The research material consisted of data collected from 64 stands of Norway spruce (*Picea abies* Karst.) and 91 stands of Scots pine (*Pinus sylvestris* L.) located in southern Finland. Both stand types contained birch (*Betula pendula* Roth and *B. pubescent* Ehrh.) admixtures. The traditional univariate approach (Model I) of using the dbh distribution (Johnson's S_B) together with a height curve (Näslund's function) was compared against the bivariate approaches, Johnson's S_{BB} distribution (Model II) and Model I_ε. In Model I_ε within-dbh-class h-variation was included by transforming a normally distributed homogenous error of linearized Näslund's function to concern real heights. Basal-area-weighted distributions were estimated using the maximum likelihood (ML) method. Species-specific prediction models were derived using linear regression analysis. The models were compared with Kolmogorov-Smirnov tests for marginal distributions, accuracy of stand variables and the dbh-h relationship of individual trees. The differences in the stand characteristics between the models were marginal. Model I gave a slightly better fit for spruce, but Model II was better for pine stands. The univariate Model I resulted in clearly too narrow marginal h-distribution for pine. It is recommended applying of a constrained ML method for reasonable dbh-h relationship instead of using a pure ML method when fitting the S_{BB} model.

Keywords parameter prediction, dbh and height distribution, Johnson's S_{BB} distribution, Näslund's height curve, *Picea abies*, *Pinus sylvestris*
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1 Introduction

The empirical breast-height-diameter (dbh) distribution is usually not determined in standwise inventories. In order to use tree-specific models in growth simulators, the dbh distribution and tree heights have to be predicted as a function of stand characteristics. Successfully applied probability-density functions (pdf) for describing dbh distributions include the Weibull function (Bailey and Dell 1973, Rennolls et al. 1985, Kilkki and Päivinen 1986), the beta-function (Päivinen 1980, Kou 1982, Siipilehto 1988, Maltamo et al. 1995), and Johnson's S_B function (Hafley and Schreuder 1977, Tham 1988, Zhou et al. 1996, Siipilehto 1999). The tradition in Finland has been to apply basal-area-weighted distribution models while elsewhere dbh-frequency distribution models have been widely used (Gove and Patil 1998). Some recent work carried out in Finland has concentrated on developing more flexible methods for describing dbh distributions, for example the percentile prediction method (Maltamo et al. 2000) and calibrating of the predicted parametric distribution (Kangas and Maltamo 2000). One advantage of these methods lies in their ability to describe the bi- and multimodality of the distribution.

When the dbh distribution is applied to a stand, individual trees can be selected randomly or systematically, the latter being the common practice. Systematic sampling based on an equal basal-area interval has been more effective than sampling based on an equal diameter interval (Kilkki et al. 1989). The less the number of sampled trees is, the greater the difference, favouring the basal-area sampling.

The more sophisticated the tree-specific growth models are, the more detailed and reasonable the predicted stand structure should be. The social status of a tree, which reflects on its further development (i.e. growth and mortality), depends not only on its relative diameter, but also on its relative height in a stand. In addition, knowledge of the height variation, both between and within dbh classes, improves the chances of successfully imitating different types of thinnings (Hafley and Buford 1985). Stand structure in terms of tree heights and diameters, including within dbh class height variation, can be described using bivariate pdf. Johnson's S_{BB} distribution has been used for this purpose in a number of studies (Hafley and Schreuder 1977, Hafley and Buford 1985, Siipilehto 1996, Tewari and Gadow 1997, Tewari et al. 1999). The trivariate S_{BBB} distribution approach has been applied in describing the joint distribution of tree diameters, heights and volumes (Schreuder et al. 1982a, Schreuder et al. 1982b). Kilkki and Siitonen (1975) presented a bivariate model based on the beta dbh distribution and Näslund's height curve together with conditional height distributions described using the beta function. No other available or generated bivariate generalizations of the univariate lognormal, gamma or Weibull distribution has been able to provide reasonable diameter-height relationships (Schreuder and Hafley 1977).

Traditionally, stand structure has been described by predicting diameter distribution together with the diameter-height relationship to estimate the average height per dbh class and hence volume (Päivinen 1980, Clutter et al. 1983). Even if the bivariate pdf method has been applied in describing stand structure in terms of tree diameters and heights, comparisons with the traditional method, or any discussion of its availability, are lacking.

The purpose of this study was to compare methods of generating individual trees in stands, including between and within dbh-class height variation. This was done by predicting the parameters of both marginal distributions and the correlation coefficient for the bivariate pdf model, and alternatively the height curve including error structure when using the traditional approach. Methods were compared in terms of the obtained stand variables, such as total stem number with total and timber-assortment volumes. The goodness of fit of the models was tested statistically and visually i.e. plotting randomly selected trees from the predicted model against observed trees. Two tree species were studied, the shade-tolerant Norway spruce (*Picea abies* Karst.) species and the shade-intolerant Scots pine (*Pinus sylvestris* L.) species.

2 Material and Methods

2.1 Data

Study material consisted of 64 stands dominated by Norway spruce and 91 stands dominated by Scots pine both with birch (*Betula pendula* Roth. and *B. pubescent* Ehrh.) admixtures (Table 1). The stands were located in southern and south-eastern Finland, respectively. The number of coniferous trees in the modelling data varied between 32 and 122 spruces and between 8 and 63 pines per plot. The diameter and the height of all the trees on the plots were measured. Distributions were computed and studied only in the case of conifers. However, the correlations between distribution parameters and the proportion of birch admixture were checked during model construction. For more detailed description of the data, see (Mielikäinen 1980, Mielikäinen 1985, Siipilehto 1999).

The models were tested using independent data that was a sub-sample of the 7th National Forest Inventory (NFI7) in Finland, the target population being the well to moderately managed, undamaged and one-storeyed stands. Each NFI7-based permanent INKA sample plot consisted of a cluster of three circular plots within a stand. In addition, each plot was divided into two parts; a longer radius for tallied trees and a shorter for more detailed measurements. The total number of tallied trees was about 120. In each plot, a smaller sub-plot was delineated with an area of one-third of the total plot area. In the sub-plot tree heights were also measured (Gustavsen et al. 1988). Diameters and heights in each data set were measured to an accuracy of 1 mm and 1 dm, respectively. Only trees on the sub-plots with measured diameters and heights were used. The stand characteristics were calculated from these observations (Table 1). The tree volumes were calculated using models with tree diameter and height as the predictors of the stem volume (Laasasenaho 1982).

2.2 Estimating the Models

2.2.1 Bivariate Johnson's S_{BB} Distribution

Bivariate Johnson's S_{BB} function (1) is based on the bivariate normal distribution (Johnson 1949). The original variables, diameters and heights were

Table 1. The mean stand characteristics for modelling and test data. There were 64 and 112 stands of Norway spruce and 91 and 103 stands of Scots pine in the modelling and test data, respectively. The meaning of shape index (ψ) is described in the model construction section.

		d_{gM}	h_{gM}	G	N	ψ
Modelling data						
Spruce	Mean	20.2	17.6	16.5	1017	0.708
	Sd	6.0	4.8	5.7	796	0.120
	Min	9.6	7.3	6.5	217	0.375
	Max	32.8	25.2	30.8	3184	0.937
Pine	Mean	25.0	21.9	13.7	359	0.867
	Sd	4.1	2.8	3.3	182	0.067
	Min	14.6	10.8	6.6	104	0.667
	Max	36.1	29.1	21.9	1100	0.985
Test data						
Spruce	Mean	21.7	18.0	19.9	782	0.801
	Sd	4.9	3.6	5.5	438	0.110
	Min	11.8	10.5	8.1	265	0.527
	Max	34.3	28.2	35.4	2925	1.034
Pine	Mean	20.7	17.0	18.1	749	0.897
	Sd	5.5	3.7	4.8	454	0.085
	Min	11.7	10.3	5.7	81	0.642
	Max	35.0	25.8	31.6	2210	1.084

transformed to standard normal variates using Formula 2.

$$P(z_d, z_h) = \left[2\pi\sqrt{1-\rho^2}\right]^{-1} \exp\left\{-0.5\left(1-\rho^2\right)^{-1}\left(z_d^2 - 2\rho z_d z_h + z_h^2\right)\right\} \quad (1)$$

in which

$$z_d = \gamma_d + \delta_d \ln\left(\frac{d-\xi_d}{\lambda_d + \xi_d - d}\right)$$
$$z_h = \gamma_h + \delta_h \ln\left(\frac{h-\xi_h}{\lambda_h + \xi_h - h}\right) \quad (2)$$

γ and δ are shape parameters, ξ is the minimum and λ is the range of either diameters (d) or heights (h), while z_d and z_h are the standard normal variates and ρ is the correlation coefficient between them.

The S_{BB} distribution was applied as a basal-area-weighted diameter and height distribution. The maximum likelihood (ML) estimates for the parameters were solved with an iterative FORTRAN program by maximizing the log-likelihood function (3) separately for dbh and h distributions (see Johnson 1949, Schreuder and Hafley 1977).

$$\ln L = -\frac{G}{2}\ln(2\pi) + G\ln\delta_i + G\ln\lambda_i - \sum_{j=1}^{n} g_j \ln(x_{ij} - \xi_i) - \sum_{j=1}^{n} g_j \ln\left(\lambda_i + \xi_i - x_{ij}\right) - \frac{1}{2}\sum_{j=1}^{n} g_j \left[\gamma_i + \delta_i \ln\frac{x_{ij} - \xi_i}{\lambda_i + \xi_i - x_{ij}}\right]^2 \quad (3)$$

where, $x_{1j} = d_j$, $x_{2j} = h_j$, $i = 1, 2$, $j = 1,..,n$; n is the number of observed diameters and heights (or classes) in a stand, x_{ij} is the observed diameter or height, g_j is the corresponding basal area of a tree, and G is the total basal area.

The iterative method was simplified by fixing the minimum diameter (ξ_d) to have the value 0 cm and minimum height (ξ_h) equal to breast height, 1.3 m. The parameters for both the marginal distributions were searched iteratively by increasing the value of the λ parameters step by step starting from the observed range. If both end points could initially be fixed, the ML estimates would have a closed-form solution (Schreuder and Hafley 1977) and also, methods using percentiles would be simple (Knoebel and Burkhart 1991). The parameters were solved as in the study of Schreuder and Hafley (1977) with the exception of basal-area-weighting. The shape parameters γ and δ were solved using Formulas 4 and 5.

$$\hat{\gamma}_i = -\bar{f}_i / s_i \quad (4)$$

and

$$\hat{\delta}_i = 1 / s_i \quad (5)$$

in which

$$s_i = \sqrt{\sum_{j=1}^{n} g_j \left(f_{ij} - \bar{f}_i\right)^2 \Big/ G},$$

$$g_j = \tfrac{\pi}{4}\left(d_j / 100\right)^2,$$

$$G = \sum_{j=1}^{n} g_j,$$

$$f_{ij} = \ln\left(\frac{x_{ij} - \xi_i}{\lambda_i + \xi_i - x_{ij}}\right)$$

and

$$\bar{f}_i = \sum_{j=1}^{n} g_j f_{ij} \Big/ G$$

When the solutions for both marginal distributions were found, the correlation parameter was calculated using Formula 6.

$$\hat{\rho} = \sum_{j=1}^{n} g_j z_{dj} z_{hj} / G \quad (6)$$

One of the properties of interest is the regression relationship between the diameter and height obtained from S_{BB}. The usual mean regression is complicated, but the median regression takes a much simpler form (Schreuder and Hafley 1977), as shown by Formula 7.

$$h = \lambda_h \theta \left[\left(\frac{\xi_d + \lambda_d - d}{d - \xi_d} \right)^{\phi} + \theta \right]^{-1} + \xi_h \qquad (7)$$

in which ϕ and θ can be denoted in terms of the S_{BB} parameters as

$$\phi = \rho \delta_d / \delta_h$$

and

$$\theta = \exp\{(\rho \gamma_d - \gamma_h) / \delta_h\}$$

The regression curve (7) can have various forms depending on the relationship between the parameters ϕ and θ (Fig. 1). The typical sigmoid form of the height curve is obtained if both parameters, ϕ and θ, are greater than one. If parameter ϕ equals one, then parameter θ should be greater than one to result in a concave form of the height curve.

To avoid unreasonable height curves, Schreuder and Hafley (1977) recommended constraining ϕ to be greater or equal to one while fitting the distribution. If necessary, this was done iteratively by increasing the range of the diameters (in steps of 0.5 cm) and by decreasing the range of the heights (in steps of 0.2 m). In this way, the parameter ϕ was increased more effectively than

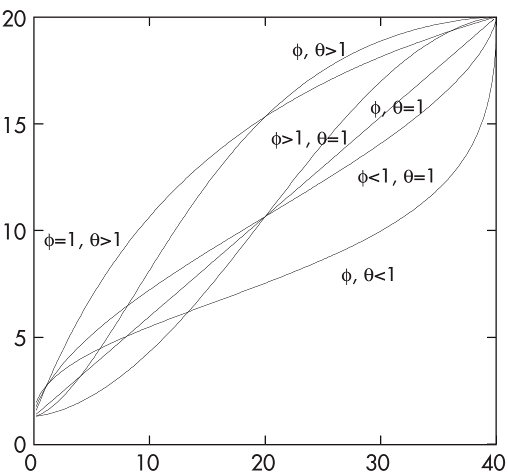

Fig. 1. The S_{BB} median regressions ('height curves') with different values of parameter ϕ and θ. (Siipilehto1996)

increasing only the range of the diameters as was used by Schreuder and Hafley (1977). Both, unconstrained and constrained solutions for S_{BB} parameters were studied in this paper.

The shape of the height distribution, conditional for diameter, changes depending on the diameter (see Siipilehto 1996). It is symmetric with respect to the median diameter. Decreasing diameter has the effect of making the conditional height distribution increasingly positively skewed (with the tail towards the higher trees). Increasing the diameter causes the distribution to become more negatively skewed.

2.2.2 Näslund's Height Curve

Näslund's (1936) height curve (8) was fitted in the linearized form (9)

$$h = \frac{d^{\alpha}}{(\beta_0 + \beta_1 d)^{\alpha}} + 1.3 \qquad (8)$$

and

$$\frac{d}{(h - 1.3)^{\alpha^{-1}}} = \beta_0 + \beta_1 d + \varepsilon_z \qquad (9)$$

in which β_0, β_1 and α are the parameters of the model. ($\alpha = 2$ for pine and $\alpha = 3$ for spruce)

The values for the power α were iteratively found by grid searches. The power of 2 worked well with pine, but the power of 3 gave considerably better fits with spruce due to increased flexibility.

The residual variation ($s_{\varepsilon z}$) of ε_z from Equation 9 was assumed to be homogenous and normally distributed (see Näslund 1936 p. 52). Applying the height model, it was transformed to concern real within-dbh-class height variation ($s_{\varepsilon h}$). Using Taylor's series expansion, the variance of the height model can be written in terms of the transformation function (z) and residual variance ($\sigma^2_{\varepsilon_z}$) as (e.g. Lappi 1993):

$$\sigma^2_{\varepsilon_h} = (g'(z))^2 \sigma^2_{\varepsilon_z} \qquad (10)$$

where

$g'(z) = -\frac{\alpha\, d^{\alpha}}{z^{\alpha+1}}$ is the first derivative of $g(z)$ and

$g(z) = \left(\frac{d^\alpha}{z^\alpha} \right) + 1.3$ is the height expressed as a function of the transformation (z)

$$z = \frac{d}{\sqrt[\alpha]{h - 1.3}}$$

Thus, standard error for predicted height is simply

$$\sigma_{\varepsilon_h} = \sqrt{g'(z)^2}\,\sigma_{\varepsilon_z} \qquad (11)$$

Substituting the first derivative $-\frac{\alpha\, d^\alpha}{z^{\alpha+1}}$ into the equation (11) and expressed as sample standard error ($s_{\varepsilon h}$) for predicted height ($\hat{h}$) yields

$$s_{\varepsilon_h} = s_{\varepsilon_z} \frac{\alpha \left[\left(\hat{h} - 1.3 \right)^{\frac{\alpha+1}{\alpha}} \right]}{d} \qquad (12)$$

in which $\alpha = 2$ for pine and 3 for spruce.

The average forms of the height curves and error variations for spruce and pine are given in Fig. 2. The height curve for pine typically bends more than that of spruce. The residual variation around the height curve first increased with increasing diameter and then the variation starts to slightly decrease with the bigger diameter classes. The standard error for pine seemed to be greater than that for spruce, particularly within the smallest dbh classes.

2.3 Model Construction and Evaluation

2.3.1 Approach

Two main approaches for predicting stand structure were studied. The traditional approach using dbh distribution together with height curve is denoted by Model I. The bivariate S_{BB} pdf for the joint dbh-height distribution is denoted by Model II. Both approaches were applied in two different ways. Either denotation Model I_ε or Model I is used, whether or not the error variation around the height curve is included in the model. Denotations Model II_ϕ or Model II are used to describe the prediction models for constrained ($\phi > 1$) or unconstrained solution for the S_{BB} distribution parameters, respectively. The predicted S_B dbh distributions in Model I are also included in Model II_ϕ for Norway spruce and in Model II for Scots pine.

Models for Johnson's S_B dbh distribution were previously presented by Siipilehto (1999). These models were intended to be used together with

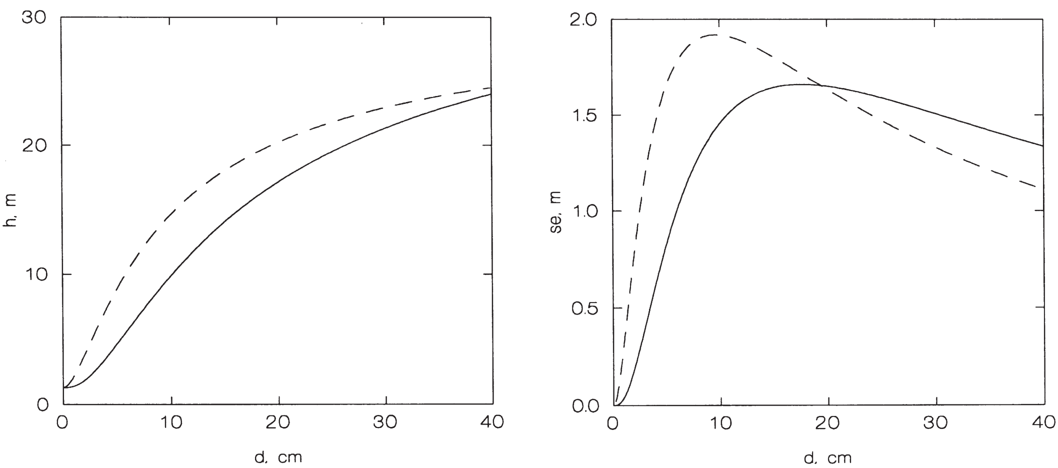

Fig. 2. Näslund's height curves (left) with the average parameters for spruce (—) and pine (- - -). Corresponding average standard error of height (right) as a function of diameter for spruce (—) and pine (- - -). Used parameters were β_0 = 0.894, β_1 = 0.185 and standard error s_{ε_z} = 0.199 with power α = 2 for pine, and β_0 = 1.811, β_1 = 0.308 and standard error s_{ε_z} = 0.277 with power α = 3 for spruce.

the new height-distribution models formulated in this study to define the bivariate S_{BB} distribution. Due to the peculiar behaviour of diameter-height relationships derived for spruce in 25 % of cases, new (constrained) diameter distributions were fitted and modelled; only a few iterative steps were needed for these constrained solutions. Thus, new models for spruce were so close to the previous models (Siipilehto 1999) that comparing them was unnecessary.

However, the previous models for Scots pine (Siipilehto 1999) were studied together with the new models. This was done for three reasons: (i) only 16 cases out of the 91 ML-estimated (unconstrained) distributions produced reasonable diameter and height relationships with respect to the value of parameter ϕ, (ii) forcing $\phi > 1$ resulted in such parameters for the marginal distributions that were hardly as well correlated to stand characteristics as the unconstrained ML parameters, (iii) the log-likelihood of the new distributions were 'far' from the maximum. It was obvious that the lack-of-fit in the height curve, together with reliable marginal distributions, and on the other hand, the lack-of-fit in the new marginal distributions, together with a satisfying diameter-height relationship, had to be compared.

2.3.2 Prediction Models

Models for the parameters γ, δ and ρ of the S_{BB} distribution, as well as the parameters β_1 of Näslund's height curve and the residual error variation (s_{ε_z}), were fitted applying least squares linear regression estimation. This was done using REG procedure in SAS (SAS 1985). The parameters were predicted with observed stand characteristics (tree-species specific basal area (G), stem number (N), basal area median diameter (d_{gM}) and height (h_{gM})).

While fitting the prediction models, observations when the parameter λ_d was greater than 100 or when the parameter δ_d was less than 0.7 were excluded. This was done because these distributions either did not converge or they indicated bimodal basal-area distributions. In addition, such distribution parameters (outliers) would violate the fit of the regression model. Altogether, seventeen pine stands were excluded for Model II$_\phi$ and five pine stands in the case of Model II. Due to the difference in size of the modelling data sets, the accuracy of the prediction models was not strictly comparable. However, none of the stands were excluded from modelling the parameters for spruce or when evaluating fitted prediction models.

Some of the parameters were solved in terms of the known median and the predicted parameters. When predicting the S_B marginal distributions, the parameter γ_i was solved according to the basal area median diameter d_{gM} or height h_{gM} using Formula 13. Thus, known medians were set for predicted marginal distributions.

$$\hat{\gamma}_i = \hat{\delta}_i \ln\left(\hat{\lambda}_i + \hat{\xi}_i - M_i\right) - \hat{\delta}_i \ln\left(\hat{\xi}_i - M_i\right) \tag{13}$$

where $i = d$, h, and M is the median

The predicted height curve was forced to pass through the known point of d_{gM}, h_{gM} by using the value of parameter β_0 given by Formula 14.

$$\hat{\beta}_0 = \frac{d_{gM}}{\left(h_{gM} - 1.3\right)^{\alpha^{-1}}} - \hat{\beta}_1 d_{gM} \tag{14}$$

Transformations in the dependent variables were used in homogenization of the residual variance, in linearization and in determining the logical behaviour of the models. For example, Fisher's z-transformation for correlation coefficient ($z_\rho = 0.5 \ln[(1 + \rho)/(1 - \rho)]$) made the linear regression model applicable and $\ln\delta$ ensured the positive value for δ. Note, the bias correcting factor ($s_e^2/2$) should be used when applying models for $\ln\delta$.

An additional stand characteristic was derived to describe the shape of the empirical diameter distribution. In advanced stands, the number of stems is not evaluated in the current forest management planning fieldwork. Consequently, the great variation in the shape of distributions can not be predicted (Siipilehto 1999). In this paper, stem number data was assumed to be known and the following shape index was utilized (15). The 'calculated basal area' ($g_M N$) was compared with the observed basal area G. The shape index (ψ) behaviour is discussed in more detail in a study by Siipilehto (1999).

$$\psi = G / g_M N \tag{15}$$

in which

$$g_M = \frac{\pi}{4}\left(d_{gM}/100\right)^2$$

2.3.3 Assessment of Model Fit

The fits of the predicted distributions were studied in many different ways. Of course, the stand total and timber-assortment volumes, as well as the stem number obtained from predicted distributions, are very important factors. These were obtained using numerical integration of the predicted univariate or bivariate models. One centimeter and half meter steps were applied for diameters and heights, respectively. In the case of Model I_ε, eleven height observations were taken systematically from a conditional height distribution.

The test criteria, relative bias (%), and standard deviation of the prediction errors (s_b), were calculated as shown in Formulas 16 and 17. The denominator in Formula 16 was 'observed stand characteristic', because otherwise the same absolute value of under- or over-estimate would have a different relative value.

$$bias_{\%} = 100\frac{1}{n}\sum_{i=1}^{n}\left[\left(Y_i - \hat{Y}_i\right)/ Y_i\right] \qquad (16)$$

$$s_b = \frac{1}{n}\sum_{i=1}^{n}\sqrt{\left(e_i - bias_{\%}\right)^2} \qquad (17)$$

in which Y_i is the observed and $\hat{Y}_i$ is the predicted stand characteristic and e_i is the relative prediction error (%) in stand i.

The fit of the marginal distributions for diameters and heights was examined using the Kolmogorov-Smirnov (KS) test at alpha 0.1 level. Because the differences between compared models could be marginal, they were additionally ranked with the KS quotient, which is the limit KS value divided by the actual KS value (Tham 1988). In the case of a great number of observations ($n > 100$) the approximative limit value was calculated (Sokal and Rolf 1981).

In addition, as conditional height distributions were used to describe within the dbh-class height variation, the forms of predicted trees should be examined. Neither KS tests for marginal distributions, nor the generated stand characteristics could discover possible irrelevance in individual tree dimensions. The fit of the predicted diameter-height relationship and the individual tree form was studied visually. Fifty trees per stand were generated from random numbers to illustrate the basal-area-weighted sample from the original stands. The generated trees were plotted together with the observed trees to enable visual evaluation of the goodness of fit. The trees forming the predicted stand plots were generated using the same random numbers to ease comparisons between the models. Evenly-distributed random numbers were transformed into standard normally-distributed random numbers by the method of Box-Muller (Press et al. 1992).

Tree slenderness (h/dbh) was used to study tree form variation. For this, the ranges for the reasonable tree slenderness were set in accordance with the modelling data. The least-slender tree form was constant for both species, Min(h/dbh) = 0.5, while the most-slender tree form was given as a function of diameter, namely Max (h/dbh) = 1.8 – 0.026(dbh for Norway spruce (see Fig. 5a), and Max(h/dbh) = dbh/(0.75 + 0.155dbh)^2 for Scots pine (see Fig. 7a). The function for Scots pine was derived from Näslund's height curve. The predicted trees outside these ranges were considered outliers.

3 Results and Discussion

3.1 Parameter Prediction Models

The estimated prediction models are given first for Model I (Table 2) and secondly for Model II (Tables 3 and 4). Note that the equations used for predicting dbh distribution in Model I (Table 2) are also included in Model II (Table 3). Contrary to the presupposed poorer degree of determination for Model II_ϕ in comparison to Model II, there was actually a slight increase. The degree of determination for the parameter λ_h for Scots pine was greatly increased due to the requirement $\phi > 1$ (Tables 3 and 4). This was due to the iterative decrease in the value of parameter λ_h, which was finally close to the observed range of heights.

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Table 2. Estimates (and standard deviations) for the prediction equations in Model I and Model I_ε.

	β_1		s_{ez}		λ_d		$\ln \delta_d$	
Spruce				Model I				
Const.	0.4097	(0.008)	–0.248	(0.095)	–12.804	(8.891)	–1.089	(0.142)
d_{gM}	0.00389	(0.001)	0.00494	(0.001)	1.096	(0.240)		
h_{gM}	–0.0102	(0.001)						
d_{gM}/h_{gM}			0.209	(0.084)				
ψ					43.572	(11.95)	2.139	(0.198)
$1/\psi$			0.127	(0.041)				
r^2	0.754		0.42		0.416		0.654	
s_e	0.016		0.064		11.12		0.425	
Pine				Model I				
Const.	0.291	(0.009)	0.0629	(0.050)	–15.405	(11.070)	–1.834	(0.337)
d_{gM}	0.00134	(0.0005)			0.876	(0.208)		
h_{gM}	–0.00634	(0.0008)						
h_{gM}/d_{gM}			0.112	(0.044)				
ψ					39.478	(12.432)	2.842	(0.387)
r^2	0.61		0.07		0.279		0.377	
s_e	0.011		0.042		7.837		0.249	

Note: dbh distribution models for Norway spruce are also included into Model II_ϕ and for Scots pine into Model II.

Table 3. Estimates (and standard deviations) for the prediction equations in Model II_ϕ for Norway spruce and in Model II for Scots pine.

	λ_h–1.3		$\ln \delta_h$		z_ρ	
Spruce			Model II_ϕ			
Constant	–5.525	(4.882)	–4.925	(1.622)	1.8381	(0.175)
d_{gM}					–0.02878	(0.0067)
h_{gM}	0.883	(0.128)				
$\ln h_{gM}$			–0.201	(0.087)		
h_{gM}/d_{gM}	15.087	(5.225)				
ψ			6.419	(1.690)		
$\ln (1/\psi)$			2.762	(1.099)	0.4436	(0.215)
r^2	0.641		0.644		0.482	
s_e	3.310		0.200		0.176	
Pine			Model II			
Constant	2.0107	(3.157)	2.3995	(0.569)	1.4236	(0.179)
d_{gM}					–0.02964	(0.0072)
h_{gM}	0.9703	(0.131)				
G	0.2449	(0.103)				
$1/\psi$			–1.5169	(0.489)		
r^2	0.423		0.101		0.167	
s_e	3.249		0.440		0.242	

Note: dbh distributions in Table 2 are used together with these models in order to define S_{BB} distribution.The dependent variable for ρ was transformed to $z_\rho = 0.5 \ln[(1 + \rho)/(1 – \rho)]$ for the linear model.

Table 4. Estimates (and standard deviations) for the prediction equations in Model II$_\phi$ for Scots pine.

	λ_d	$\ln \delta_d$	λ_h–1.3	$\ln \delta_h$	z_ρ
Pine			Model II$_\phi$		
Const	–45.734 (16.82)	–1.477 (0.330)	1.580 (1.311)	–1.040 (0.485)	1.279 (0.195)
ψ	73.175 (19.06)	2.839 (0.381)		1.586 (0.560)	
d_{gM}	1.389 (0.352)				–0.022 (0.008)
h_{gM}			1.072 (0.060)		
r^2	0.355	0.435	0.817	0.100	0.104
s_e	10.97	0.225	1.322	0.331	0.248

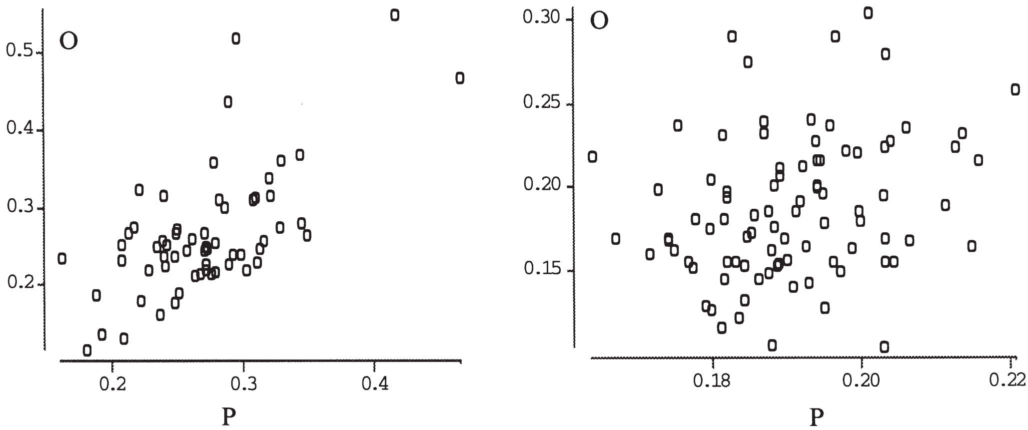

Fig. 3. Predicted (P) and observed (O) standard error (s_{ε_z}) of linearized Näslund's height curve for spruce (left) and for pine (right).

correlation parameter (ρ), as well as for the error variation (s_{ε_z}) for Scots pine were rather low (7–17 %), considerably lower than for spruce (42–48 %) (Tables 2, 3 and 4, Fig. 3). This could be partly due to the generally lower correlation between diameter and height for pine when compared to spruce. For example, the Pearson correlation coefficient between diameter and height was 0.95 for spruce and 0.79 for pine for the entire data sets, while the standwise correlations varied within the range 0.79–0.98 and 0.21–0.89, respectively.

Table 5. The number of predicted Norway spruce stands that did not pass the KS one-sample goodness-of-fit test at alpha 0.1 level. The total numbers of stands were 64 in the modelling data and 112 in the test data.

Model	Model II$_\phi$		Model I$_\varepsilon$	Model I
Distribution	dbh	h	h	h
Modelling data	2	1	2	1
Test data	2	1	1	1

3.2 Model Evaluation for Norway Spruce

3.2.1 Marginal Distributions

According to the results of the KS tests (α = 0.1), the predicted distributions fitted well with the observed distributions and the differences between the models were marginal (Table 5). The derived height-frequency distributions did not pass the KS test in one or two cases in Model I or Model I$_\varepsilon$, respectively. According to the KS quotient (see Tham 1988) the predicted S_B height distribution (Model II_{ϕ}) was superior in 26 or 38 cases out of 64 to that derived from Model I_{ε} or Model I, respectively.

3.2.2 Stand Characteristics

Because the dbh distribution model was common for the spruce models, the results in stem-number estimation were the same; the bias in stem number was an overestimate of 4 % with a deviation of 8 % in the *model data*. All the models for Norway spruce proved to be relatively accurate in predicting the total and timber-assortment volumes of a stand (Table 6). The bias in total volume of spruce trees varied from almost unbiased estimates (0.05–0.11 %) (Model I and Model I_{ε}) to a bias of 0.5 % (Model II_{ϕ}). All the models slightly under-estimated the volumes of saw-timber and over-estimated those of pulpwood. The relative errors in saw-timber volumes were quite high (22–24 %) even though the smallest saw-timber fractions (< 5 m^3ha^{-1}) were ignored during the computing of relative errors. Näslund's height curve proved to generate slightly more accurate assortment volumes than the bivariate S_{BB}, but the differences between the models were marginal. Due to the clearly bimodal empirical distributions of the two spruce stands, the predicted distributions did not fit and lead to a clear underestimation (22 and 43 m^3ha^{-1}) of the saw-timber volume. These two stands are not included in Table 6.

In the *test data* the total and saw-timber volumes were under-estimated between 3.0 % and 6.6 %, and the smaller fractions, pulpwood and non-industrial wood, were correspondingly over-estimated (Table 7). The smallest biases and error deviations were most often given by models including Näslund's height curve. There were

Table 6. The absolute and relative bias (and standard deviation) of the prediction errors in the total (V_T), saw-timber (V_S), pulpwood (V_P), and non-industrial wood (V_N) volumes for Norway spruce in the *modelling data*. The smallest relative biases and deviations are highlighted in **bold**.

Model	V_T, m^3ha^{-1}	V_S, m^3ha^{-1}	V_P, m^3ha^{-1}	V_N, m^3ha^{-1}
Model II_{ϕ}	0.98 (3.90)	2.50 (5.95)	–0.74 (4.54)	–0.41 (0.79)
Model I_{ε}	–0.25 (3.14)	0.99 (5.63)	–0.60 (4.41)	–0.39 (0.71)
Model I	–0.23 (3.19)	0.97 (5.57)	–0.52 (4.41)	–0.44 (0.70)
	%	%	%	%
Model II_{ϕ}	0.54 (2.59)	2.19 **(21.71)**	–2.91 (8.89)	–7.28 (8.68)
Model I_{ε}	–0.11 **(2.20)**	**1.34** (23.82)	–2.14 (8.40)	**–5.93 (8.25)**
Model I	**–0.05** (2.23)	1.60 (23.29)	**–1.94 (8.34)**	–6.73 (8.39)

Table 7. The absolute and relative bias (and standard deviation) of the prediction errors in total (V_T), saw timber (V_S), pulpwood (V_P), and non-industrial wood (V_N) volumes for Norway spruce in the *test data*. The smallest relative biases and deviations are highlighted in **bold**.

Model	V_T, m^3ha^{-1}	V_S, m^3ha^{-1}	V_P, m^3ha^{-1}	V_N, m^3ha^{-1}
Model II_{ϕ}	5.32 (5.72)	5.71 (8.31)	–0.19 (8.31)	–0.19 (0.47)
Model I_{ε}	4.47 (5.48)	3.81 (7.64)	0.80 (8.00)	–0.14 (0.40)
Model I	4.54 (5.51)	3.64 (7.73)	1.08 (8.11)	–0.18 (0.43)
	%	%	%	%
Model II_{ϕ}	3.29 **(3.46)**	**5.16** (19.77)	–2.08 (11.90)	–4.42 (11.03)
Model I_{ε}	**3.04** (3.56)	6.56 (20.2)	**0.68 (10.35)**	**–2.54 (10.09)**
Model I	3.12 (3.63)	6.11 **(19.25)**	1.11 (10.52)	–3.45 (10.10)

hardly any differences in the assortment volumes regardless of whether the residual error term was included (Model I_ε) or excluded (Model I) when the Näslund's height curve was applied.

3.2.3 Results in Visual Assessment

When the generated trees were plotted together with the observed trees in order to visually evaluate the goodness of fit, no really peculiar tree was to be found. In a few cases, the predicted height curve did not coincide with the observed trees very well. In these cases, the height curve was concave or almost linear instead of being a better fitting sigmoid (see Fig. 4).

Perhaps the best way to characterise the overall usefulness of the various models was through an analysis of acceptable stem form described by slenderness (height/dbh). The predicted trees outside the set ranges, given in paragraph 2.3, were considered to be outliers and they were highlighted with larger symbols (Fig. 5b). There seemed to be two trees that were too slender and eight that were too tapered in the modelling data in accordance with set criterions (Fig. 5a). Only a few more outliers were found within the data of predicted trees. Two trees out of 3 200 randomly-selected predicted trees were clearly too tapered and ten trees (each in a different stand plot) too slender when using the bivariate distribution (Model II_ϕ). In general, just 0.4 % of the predicted trees had unreasonable tree forms. Despite the poor fit of the regression in stand 62 (see Fig 4), caused by bimodality and thereby extremely low shape index (ψ), no tree form outside the set ranges was found in this stand. Generally, too wide conditional height distribution resulted in the tapered tree forms.

If the trees were generated using Model I_ε, only one tree had a visibly excessively tapered form, but the forms of 26 trees were too slender. This was partly due to the greater height variation within the largest diameter classes (dbh greater than 30 cm), partly because of the not-so-concave height curve when compared to the S_{BB} median regression including the asymptote of the greatest height. The proportion of outliers was 0.8 % and these were found in 11 stand plots out of 64 by both of the applied methods. If the tree heights were predicted using Model I (without error variation), the same kind of errors were still found.

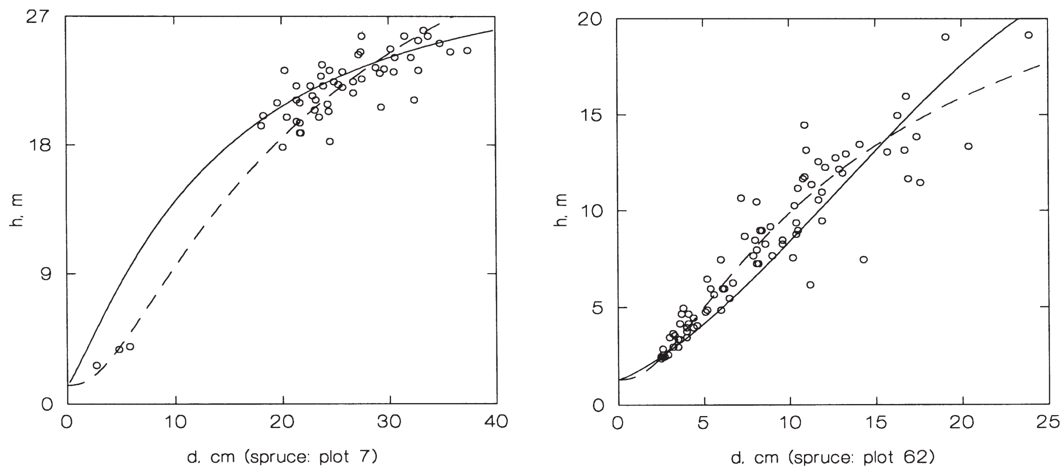

Fig. 4. Examples of spruce stand plots. The height curve, derived from predicted S_{BB} distribution in Model II_ϕ (—), was concave (left) or almost linear (right) instead of sigmoid as the Näslund's height curve in Model I (– –). Measured spruce trees are indicated in circles.

3.3 Model Evaluation for Scots Pine

3.3.1 Marginal Distributions

According to KS tests ($\alpha = 0.1$), all of the predicted basal-area distributions fitted to observed distributions and only one height-frequency distribution did not pass the KS test in the modelling data (Table 8). However, the KS quotient (Tham 1988) showed that the height distributions obtained using Model II_ϕ were superior to the distributions obtained using Model II in 70 cases out of 91. Also, both fitted better than did the height-frequency distribution from Model I_ε in 73 and 75 cases out of 91, respectively. There was only some evidence in the test data for the presumed better fit of the marginal distributions based on Model II as compared to Model II_ϕ. Finally, excluding the error variation (Model I) resulted in excessively peaked and narrow height distributions. Ultimately, 21 % of these distributions did not pass the KS test, which was about ten times more than when including the residual error term (Model I_ε).

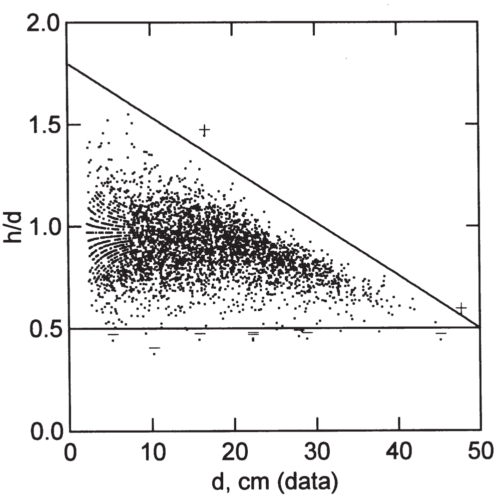

Fig. 5a. Observed tree forms (h/dbh) in the data set of Norway spruce. Lines h/dbh = 1.8 – 0.026 dbh and h/dbh = 0.5 were set for ranges for reasonable tree forms. Tree form above (+) or below (–) set ranges is considered as an 'outlier'.

Table 8. The number of Scots pine stands that did not pass the KS one-sample goodness-of-fit test at alpha 0.1 level. There were a total of 91 stands in the modelling data and 103 in the test data.

Model	Model II		Model II_ϕ		Model I_ε	Model I
Distribution	dbh	h	dbh	h	h	h
Modelling data	0	1	0	1	4	11
Test data	2	1	2	3	2	22

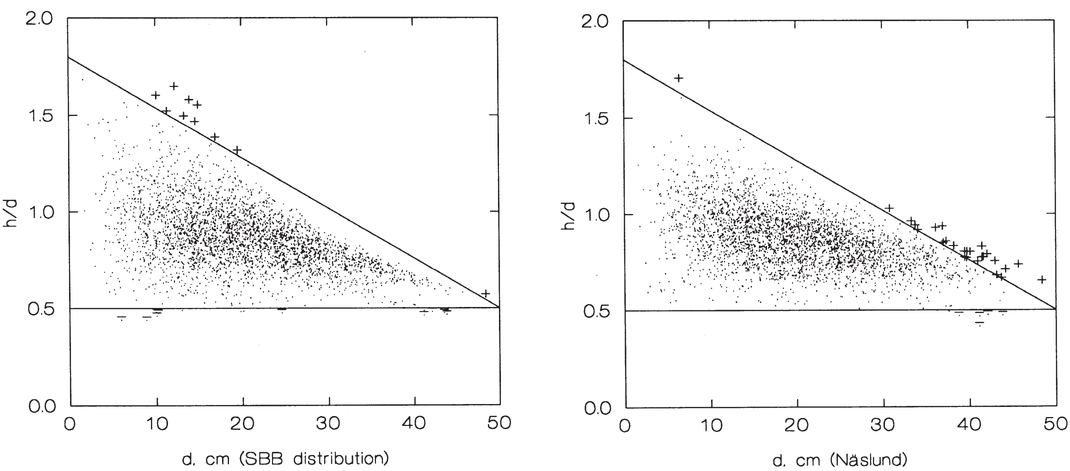

Fig. 5b. 'Outliers' (+, –) obtained using Model II_ϕ (left) and using Model I_ε (right).

Table 9. The absolute and relative biases (and standard deviations) of the prediction errors in total volume (V_T), saw-timber volume (V_S), pulpwood volume (V_P), and non-industrial wood volume (V_N) for Scots pine in the *modelling data*. The smallest relative biases and deviations are highlighted in **bold**.

Model	V_T, m^3ha^{-1}		V_S, m^3ha^{-1}		V_P, m^3ha^{-1}		V_N, m^3ha^{-1}	
Model II	–0.86	(1.43)	0.85	(3.00)	–1.52	(2.65)	–0.18	(0.26)
Model II_ϕ	0.20	(1.61)	–0.73	(3.10)	0.84	(2.38)	0.08	(0.19)
Model I_ε	–0.63	(1.34)	0.47	(3.08)	–0.98	(2.58)	–0.13	(0.19)
Model I	–1.08	(1.36)	–0.06	(3.00)	–0.89	(2.56)	–0.13	(0.18)
	%		%		%		%	
Model II	–0.69	(0.87)	0.34	(4.61)	–5.90	(9.09)	–10.54	(8.19)
Model II_ϕ	**0.11**	(1.02)	–0.78	**(3.50)**	**2.01**	**(8.06)**	**4.41**	**(6.66)**
Model I_ε	–0.51	**(0.80)**	**0.002**	(4.55)	–4.04	(8.96)	–7.40	(6.86)
Model I	–0.81	(0.81)	–0.51	(4.62)	–3.70	(8.97)	–7.39	(6.79)

Table 10. The absolute and relative biases (and standard deviations) of the prediction errors in total volume (V_T), saw-timber volume (V_S), pulpwood volume (V_P), and non-industrial wood volume (V_N) for Scots pine in the *test data*. The smallest relative biases and deviations are highlighted in **bold**.

Model	V_T, m^3ha^{-1}		V_S, m^3ha^{-1}		V_P, m^3ha^{-1}		V_N, m^3ha^{-1}	
Model II	3.31	(1.86)	4.86	(5.17)	–0.98	(5.25)	–0.57	(0.78)
Model II_ϕ	4.37	(2.06)	5.19	(5.34)	–1.04	(5.22)	0.21	(0.49)
Model I_ε	3.58	(1.93)	4.95	(5.26)	–0.91	(5.34)	–0.46	(0.60)
Model I	3.27	(1.92)	4.54	(5.26)	–0.82	(5.35)	–0.45	(0.59)
	%		%		%		%	
Model II	**2.31**	**(1.31)**	**5.07**	(26.84)	–3.64	(9.83)	–13.81	(12.82)
Model II_ϕ	3.04	(1.35)	7.63	(21.89)	**–1.51**	**(9.05)**	**4.56**	**(9.67)**
Model I_ε	2.56	(1.52)	9.80	(20.54)	–2.52	(9.32)	–11.61	(10.48)
Model I	2.45	(1.66)	8.83	**(19.18)**	–2.19	(9.28)	–11.53	(10.26)

3.3.2 Stand Characteristics

All the models were relatively accurate in predicting the stand stem number or stand total volume in *model data* (Table 9). The stem number was 2.5 % biased (with standard deviation of 4 %) such that Model II gave an over-estimate and Model II_ϕ gave an underestimate of the same size. The volume characteristics were slightly over-estimated, except in the case of Model II_ϕ, which gave the most accurate total, pulpwood, and non-industrial wood volumes. Model II gave the least accurate estimates of the smallest trees (pulpwood and non-industrial wood fractions). However, the differences between the models were marginal. The stem numbers obtained were biased with respect to the median diameter if Model II was applied.

In the *test data* the total volumes and the saw timber volumes were under-estimated between 2.3 % and 9.8 % whereas the smaller fractions, pulpwood and non-industrial wood, were correspondingly over-estimated (Table 10). The smallest biases and error deviations were again most frequently given by Model II_ϕ. If Näslund's height curve was used, the differences in timber-assortment volumes were marginal when either including (Model I_ε) or excluding (Model I) the residual error term (ε), even though the KS test results differed drastically. The total volume was given accurately by Model II, but the timber-assortments volumes were usually the most inaccurate.

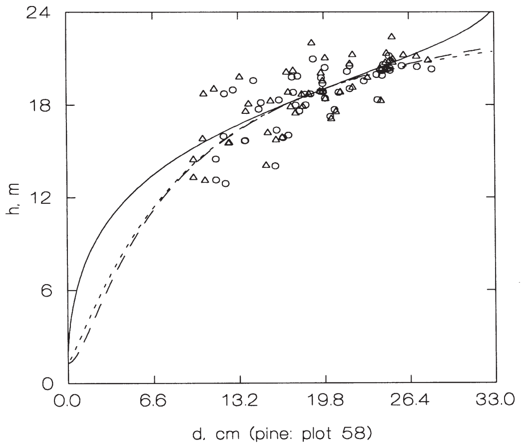

Fig. 6. An example of a pine stand plot. The height curve derived from S_{BB} in Model II (—) bended unreasonably. Predicted Näslund's height curve in Model I (– –) and also height curve obtained with Model II_{ϕ} (- - -) were reasonable and fitted well. However, trees generated from S_{BB}, either using Model II (▵) or Model II_{ϕ} (○), were quite similar.

3.3.3 Results in Visual Assessment

There were marked differences between Model II and Model II_{ϕ}, but the latter was quite similar to Model I as far as the dbh-height relationship was concerned (Fig. 6). The height curve of the Model II did not only bent unreasonably in the upper part of the distribution, but it was also too steep in the smallest diameters. Despite the great differences throughout the S_{BB} probability space in Model II and Model II_{ϕ}, the randomly-selected generated trees were quite similar (Fig. 6).

A total of eight measured Scots pines (0.22 %) in the modelling data were just outside the set ranges for tree slenderness. Model I_{ε} produced 13 slender outliers (0.28 %) out of 4 600 generated trees (Fig. 7a). Correspondigly, Model II produced 21 slender and 2 tapered outliers (0.50 %) and Model II_{ϕ} resulted in only six slender outliers and one tapered outlier (0.15 %) out of 4 600 generated trees (Fig. 7b). The 'outliers' produced applying S_{BB} or univariate S_B together with height curve were mostly to be found in the opposite parts of the dbh distribution.

4 Conclusions

No matter which approach was used, the goodness of fit of the bivariate dbh and the height distribution model was fairly good. The lack-of-fit proportion was greater than the risk level in one (univariate) case only; 21 % of the height distributions for Scots pine did not pass the KS test if Näslund's height curve was applied without error variation. This was simply due to the excessively narrow and peaked height distributions generated. However, this did not affect the stand characteristics, which were relative accurate for all the models applied. Unfortunately, there are no such bivariate prediction models to compare the results achieved in the present study generated. The accuracy achieved in stem volume was, of course, close to that presented by Siipilehto (1999) with univariate S_B model. Due to three small-sized plots in test data, the risk of a bi- or multimodal distribution was obvious.

There were no drastic differences in the goodness of fit for pine regardless of the S_{BB} model applied. Indeed, Hahn and Shapiro (1967) showed that the quality of fit is relatively unaffected by the choice of lower bound and range as long as they are consistent with the data to be fitted. The marginally improved goodness of fit of height distributions for the constrained ($\phi > 1$) model was most likely due to the generally enhanced correlation coefficient (ρ) between standard normalized diameters (z_d) and heights (z_h) and the better fit of the diameter-height relationship in some extreme cases. As regards the independent test data, the constrained model (Model II_{ϕ}) fitted better than the unconstrained model (Model II) for S_{BB} in the lower part of the distribution. In the accuracy of the total and saw-timber volumes the situation was vice versa. Two reasons may be found for this, firstly, the regression within the smallest diameters obtained with Model II was too steep and secondly, the predicted maximum end points were closer to the observed ones when using Model II compared with Model II_{ϕ}. Finally, Model II produced more unreasonable tree forms (outliers) than Model I_{ϕ}. This was not surprising as ML estimation focuses fitting the marginal distributions, not the relationship between dbh and height. Thus, the most unreasonable dbh-height relationships were derived

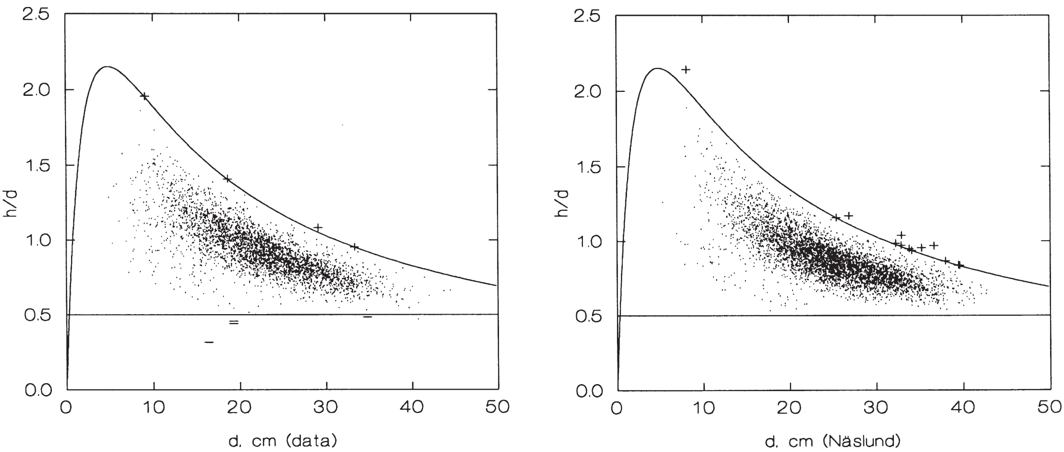

Fig. 7a. Observed tree forms (h/dbh) in the data set of Scots pine (left) and by Model I_ε (right). Line h/dbh = 0.5 and curve h/dbh = dbh/(0.75 + 0.155 dbh)^2 were set for ranges for reasonable tree form. Tree form above (+) or below (–) set ranges is considered as an 'outlier'.

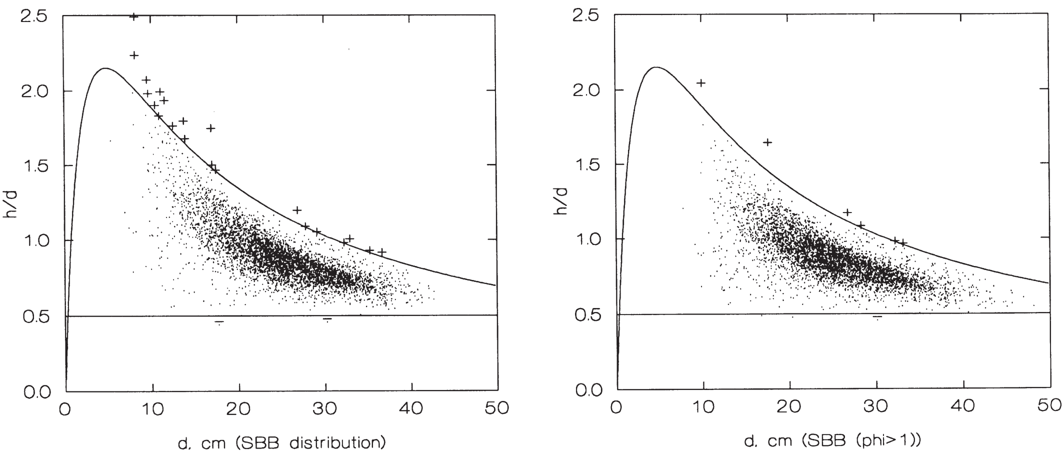

Fig. 7b. 'Outliers' (+, –) obtained using Model II (left) and using Model II_ϕ (right).

from predicted S_{BB} distribution using Model II and they resulted in the least accurate volume estimates.

Both of the principle methods, bivariate S_{BB} pdf and S_B dbh distribution with height curve and error structure, could be applied successfully in predicting the joint distribution of tree heights and diameters. The proportion of predicted outliers (i.e. tree form beyond the ranges of the modelling data) was slightly smaller when using the Model II_ϕ than when using Model I_ε, respectively 0.4 % and 0.8 % for the models for Norway spruce and 0.15 % and 0.28 % for Scots pine. Thus, setting the constraint ϕ >1 for satisfying diameter-height relationships is recommended when using the S_{BB} distribution.

The advantage of the S_{BB} model was most probably based on the asymptote of the greatest heights; there were no trees with peculiar forms in the upper part of the bivariate distribution. If the diameter-height relationship was reasonable ($\phi > 1$), extreme form values were very rarely to be found. On the contrary, the form of the predicted diameter-height relationship was not as controlled as with Näslund's height curve. The greatest advantages in using Näslund's height curve lay in the sigmoid form of the curve within the predicted variation of the parameter β_1. In addition, the standard deviation of the random error in height was reasonably dependent on tree diameter. Generally, randomly selected trees around the height-curve fitted visually better with the observed standwise data than with the applied S_{BB} distribution. The reason for excessively slender generated Norway spruces lay in the combination of the predicted dbh distribution being too wide and the almost linearly increasing height as diameter increased. Thus, these outliers were found without random error being involved.

The trees were selected in proportion to their basal area, emulating angle-count selection. Thus, the number of smaller trees was lower among the generated trees than in the modelling data. Gove and Patil (1998) presented a framework that allows the basal area-dbh distribution to be parameterized once the dbh-frequency distribution is known. However, this technique is complicated with the S_B function and it was not used. It was obvious that height variation within the smallest diameter classes was greater when using the S_{BB} model than when using Näslund's height curve with random error (see Siipilehto 1996). Consequently, if the randomly generated trees were selected in proportion to stem frequency, the results obtained when using the S_{BB} could have been slightly worse than those obtained using Näslund's height curve. Indeed, Siipilehto (1996) found more outliers when using bivariate S_{BB} frequency distribution (0.8–1.5 %) than when applying univariate S_B dbh distribution with Näslund's height curve (0.5 %). Note that when the stand characteristics in this study were computed by integrating frequency distributions, the least accurate non-industrial wood fractions were given by the unconstrained Model II.

The shape of the conditional height distribution did not change with respect to diameter, if Näslund's height curve is used; it was always a normal distribution. This was not the case with the S_{BB}. The conditional height distribution was negatively skewed, i.e. the longer tail pointed towards the smaller heights among the biggest dbh classes. Perhaps the latter was biologically more reasonable than having a normal distribution.

Johnson's S_{BB} distribution could be recommended for describing bivariate stand structure, particularly if the requirement $\phi > 1$ for reasonable diameter-height relationship is applied. In addition, it is to be recommended that tree selection be made in compliance with the basal-area interval, as this will have the effect of only a few small trees being selected. This recommendation is a consequence of the relatively wide conditional height distribution of trees falling into the smallest diameter classes increasing the risk of peculiar tree forms occurring. However, the traditional approach, together with predicted residual height variation, could be recommended as well. Some improvement of the traditional approach could be achieved by neglecting the homogenous and normally-distributed error assumptions. Williams et al. (1996) showed how the model for gross tree volume could be improved by estimating the prediction model and non-normal error structure together.

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